A THREE LEVEL ASYMPTOTIC ANALYSIS OF MINIMUM ORDER B-TREES

By

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Bachelor of Arts

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1975

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
MASTER OF SCIENCE
July, 1977

A THREE LEVEL ASYMPTOTIC ANALYSIS

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PREFACE

This thesis presents a three level analysis of the asymptotic behavior of minimum order B-trees. It is hoped that this analysis provides insight to the asymptotic behavior of B-trees.

I would like to thank my major adviser, Dr. James Van Doren, for providing the foundation for this analysis, including the instruction, guidance, and interest he gave toward its completion.

I am grateful to Dr. G. E. Hedrick III and Dr. J. P. Chandler for providing a scholastic atmosphere in which it was a pleasure to learn.

I am indebted to my parents for providing the patience, consideration, encouragement, and help in all my educational pursuits.

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CHAPTER I

INTRODUCTION

One of the ways to represent ordered lists of data is through trees. A tree is defined by Knuth (3, page 305) as a finite set of one or more nodes such that there is one root from which the other nodes descend, and each of the descendents is a tree. These descendents are called subtrees of the root. The node structure can be considered to be the key by which the list is stored, the information surrounding the key, and the pointers to the nodes descending from the given node. The pointer may be a symbolic pointer which is the value of an index, a relative pointer which is based on another address, or an absolute pointer which contains the address in memory of the item. The number of subtrees descending from a given node is the degree of that node.

B-trees are one form of trees with good search characteristics. A sample B-tree node is given in Figure 1. B-trees are uniform depth trees (6) in which all leaf nodes lie at the same level. They are of uniform depth in that the graph theoretic distance to any leaf node from the root is the same. A B-tree (1, page 15) is a tree in which every node has less than or as many descendents as the order, the root has at least two descendents, a non-leaf node with k descendents has k-1 keys, and the minimum number of keys is m-2 where m is the order.

P_o	K_1	INFO	P_1	K_2	INFO	P_2

Figure 1. A Sample B-Tree Node

The order, m, of a B-tree is the maximum number of descendents from a given node, and the maximum number of keys in a node in a B-tree of order m is m-1. Insertions occur at the bottom level and when a key is inserted into a full node, the node splits and the middle key is propagated to the next level. A more complete discussion of B-trees is given in Knuth (3) and Davis (1). Since insertions occur at the bottom level, the levels are counted from the bottom level up. The bottom level is level 1, the nodes the bottom level nodes descend from are on level 2, and the nodes above them are on level 3. A minimum order B-tree, referred to as a 2-3 tree, is one with a maximum of two keys and three descendents in a full node. Figure 2 represents a full 2-3 node. In a 2-3 tree with n keys there are n+1 external nodes at the bottom level representing null branches. The analysis presented in this thesis is based on minimum order B-trees.

P_1	K_1	INFO	P_2	K_2	INFO	P_3	K_3	INFO	P_{m-1}	K_{m-1}	INFO	P_m

Figure 2. A Full 2-3 Node

The n+1 external nodes at the bottom level can be characterized by the configuration of nodes that they descend from. The state of an external node is the type of tree which it descends from. Figure 3 represents a third level tree and there are twelve external nodes represented by triangles which reside in this state. The analysis done in this thesis determines the behavior of 2-3 trees for the asymptotic probabilities of a given state occurring. The probabilities become asymptotic upon iterating to a stable level. These probabilities can be used as a relative measure to determine the performance of 2-3 trees. Upon determining the asymptotic activity of certain classes of external nodes the results are used to compute storage, utilization, and maintenance characteristics.

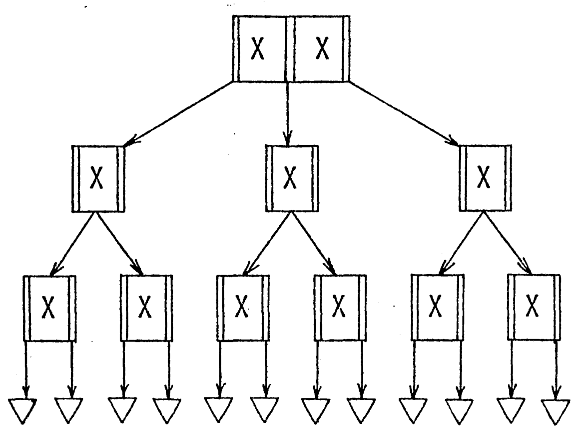


Figure 3. Sample Third Level Tree (X values represent different key values, and ▽ represent the external nodes.)

CHAPTER II

FOUNDATIONS OF THE ANALYSIS

The analysis done in the "Theoretical Foundation" is based upon the classification of external nodes according to the nodes that they descend from. The analysis determines a set of external node states which represent a particular one, two, and three level tree for one, two, and three level external node state classifications. The external nodes in Figure 4 would be classified as to the external nodes descending from the second level root. A B-tree is built from the bottom up, hence the levels are counted from the bottom up.

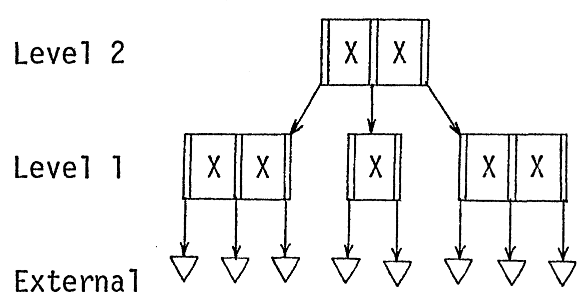


Figure 4. Example of Second Level Tree

External nodes may be classified into states according to the type of subtree from which they descend. If the asymptotic probabilities for the states of external nodes are known, the number of one key and two key nodes can be ascertained from the configuration of the trees represented by each state of external nodes. Hence, by knowing the asymptotic probabilities the utilization can be computed as well as other information related to the behavior of the trees. The utilization is defined as (5, page 21)

$$\text{Utilization} = \text{\# KEYS/\# KEY POSITIONS AVAILABLE} \tag{2.1}$$

Considering the types of nodes as the first level, the nodes can be grouped into two classes (6). There are nodes with one key with two state one external nodes descending (see Figure 5). The remaining key position is unoccupied and the pointer field is inactive. The other classification of external nodes has a two-key parent with three, state two, external nodes descending from each level one note. The frequency of splitting at the bottom level is relative to the number of insertions into the level one nodes which are full. If the relative frequencies of state one and state two external nodes are known, the utilization for level one can be computed. For each one key, level one node there are two, state one, external nodes and for each two key, level one node there are three, state two, external nodes.

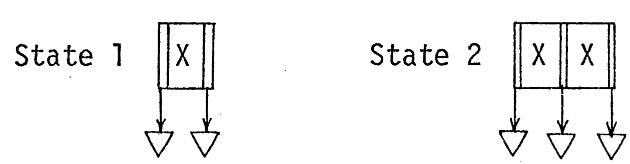


Figure 5. Two Types of Bottom Level Nodes

To analyze the second level in the following section on the "Theoretical Foundations" it is necessary to determine the possible arrangement of nodes on two levels. The 2-3 levels are counted from the bottom up (6) in this thesis and the second level external node classifications are based on combinations of second level trees. The root node of the tree from which the second level, state one external nodes descend has one key, with two first level nodes as descendents and from each there are two external nodes. Hence, there is a total of four state one external nodes. If the second level node has one key and has descendents at the first level, a two key node and a one key node, the external nodes are in state two. From Van Doren (6) the order of occurrence of the bottom level nodes is unimportant. The left branch of the second branch can be the one or two key level one node. If two second level trees are considered having five external nodes each but with one having a two key bottom level node descending from the left branch and the other has the two key descending from the right branch, the external nodes of both trees are considered to be in state two (see Figure 6). These "mirror images" are in the same state with a mapping of the larger level one node to the left subtree of the second level node and the smaller first level node to be the right branch.

If a key is inserted into a tree with state two external nodes, one of two alternatives can occur. The insertion can be made into a full bottom level node or into a one key bottom level node (see Figure 7). If the insertion occurs into the one key bottom level node, the resultant node has two keys. Hence, the insertion results in six, state three external nodes for the second level. If the insertion occurs in the subtree with the full bottom level node, a split occurs

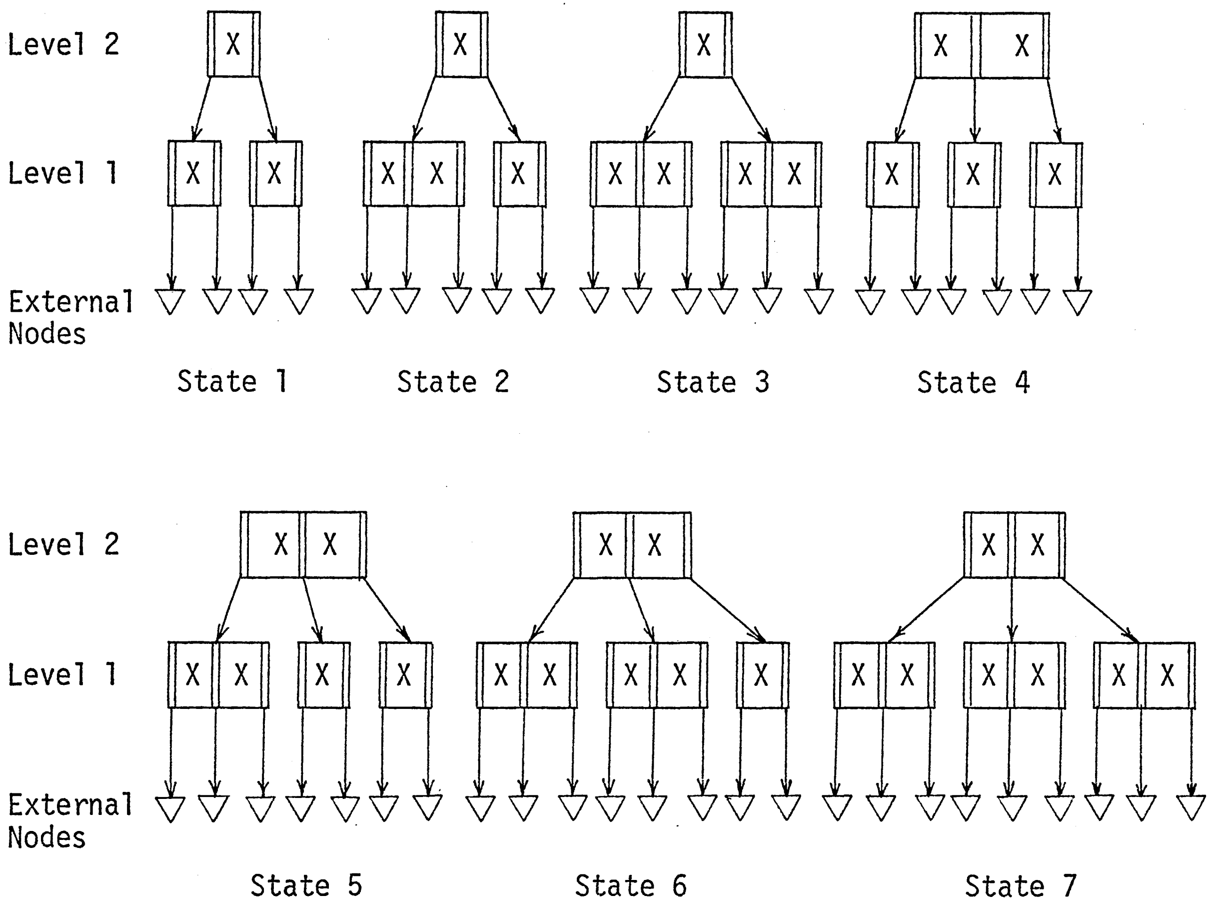


Figure 6. Second Level External Node Classifications

and a key is propagated to the second level. This forces the second level node to have two keys with three one key, bottom level descendents having a total of six external nodes. Thus a state four external node descends from a two key grandparent and a one key parent. If an insertion occurs in the tree with state three external nodes, a split is forced and the transition is made into seven state five external nodes.

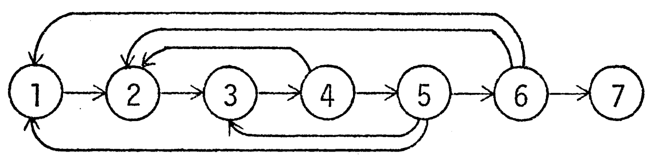


Figure 7. External Node State Transformations Occurring upon Insertion into a Given Level Two Tree (6)

Second level trees with seven external nodes are considered to be in state five. As in Figure 8 a state five second level tree is considered with the largest bottom level node to be the left subtree to avoid the redundancy of mirror images. If an insertion occurs in a state five, second level tree, as in Figure 9, two resultant trees are possible. The key could be inserted into one of the three external nodes descending from the full bottom level node or into the four external nodes descending from the two one key bottom level nodes. If the insertion occurs and two state one second level trees are created. When the insertion occurs in one of the other two first level nodes, a state six second level tree is created. The mirror image redundancy is avoided by mapping all second level trees with eight external nodes into state six. The bottom level nodes are identified with the largest on the left, the next to the largest in the middle, and the remaining nodes on the right. Insertion into a state six second level tree implies either the creation of a state one second level tree and a state two second level tree which occur when a bottom level node splits or the transition to a state seven second level tree with nine external nodes. There are seven unique classifications of second level trees. The utilization can be computed by knowing the asymptotic probabilities of occurrence of each of the seven external node states. By knowing the asymptotic probabilities of occurrence the number of keys and the number of nodes can be deduced; hence, the utilization can be computed. By knowing the relative frequencies of occurrence of each of the states the splitting characteristics can be computed by decomposing the states into one and two key nodes for the two levels.

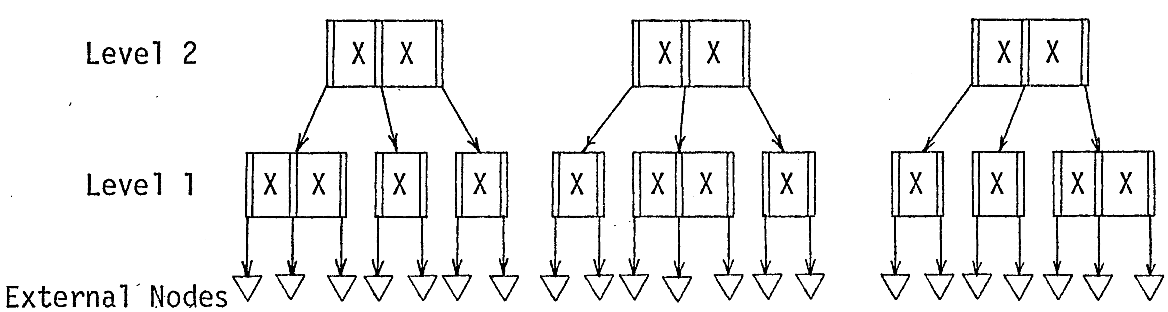
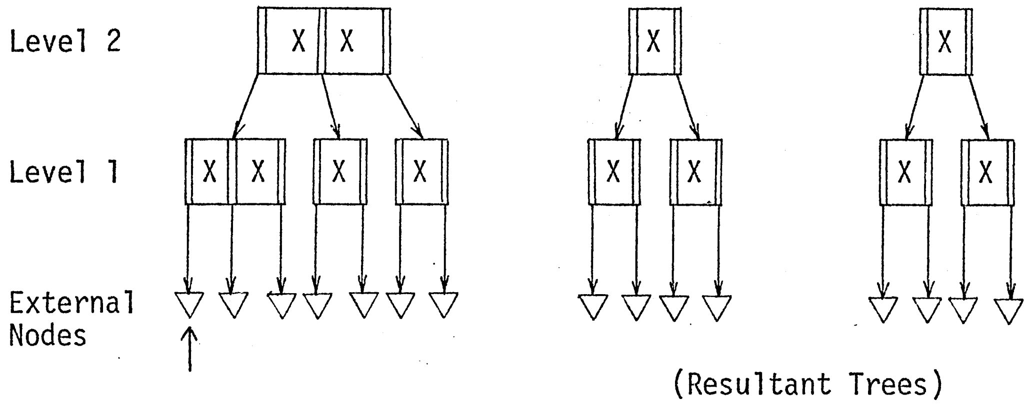
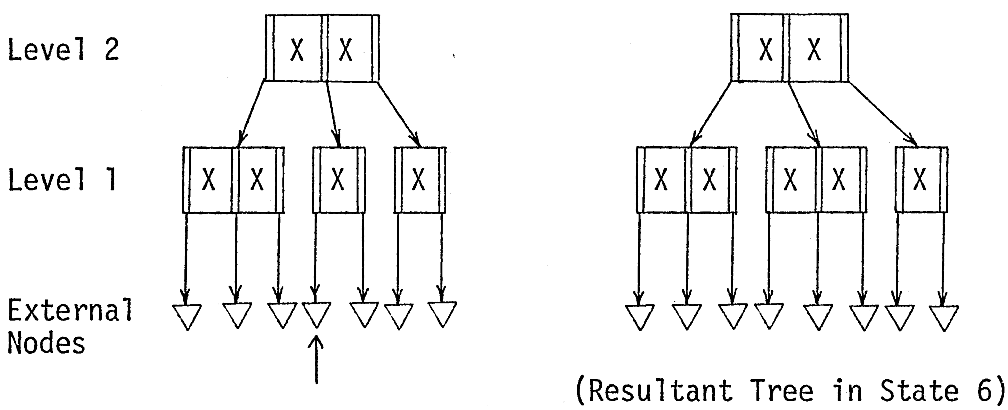


Figure 8. State Five Second Level Trees, with Representative State Five Tree on Left



State Five Second Level Tree with Insertion Position Indicated by Arrow



Insertion Symbolized for Either of the One Key First Level Nodes

Figure 9. Insertion into a Second Level Tree

Third Level State Classification

A given three level 2-3 tree may be considered to have a left subtree, a right subtree, and if there are two keys at the third level, a middle subtree. One way of classifying the external nodes descending from a third level grandparent is by considering the second level subtrees that are branched to by the left, right, and middle branches of a third level node. For the states with a grandparent with one key there are seven possible values for the left and right subtrees. The number of combinations of two objects taken from a set of seven objects where order is unimportant is

$$\frac{7!}{2!\ (7-2)!} = 21. \tag{2.2}$$

This value is incremented by the seven combinations of equal left and right subtrees which sums to 28 states descending from a grandparent with no middle key. The first 28 states are each uniquely determined with the larger subtree to be from the left branch of the third level node. The middle branch is either null or points to one of the seven combinations of two level subtrees. Since there are a total of eight possibilities for the middle branch, counting the null case, and 28 combinations for left and right subtrees, a total of 224 states $(8 \cdot 28)$ for classifying external nodes of three level trees may be used.

Theoretical Foundation

The method of analysis for the first and second levels is used as the foundation for the third level. This technique of analysis is from Van Doren (6) and Mitchell (5). To analyze the first level there is the assumption that the external nodes represent equally likely targets of insertion, a random insertion assumption. Let the probability of a state one external node by $p(n)$ for n external nodes and the probability of a state two external node which descends from a two key bottom level parent be $q(n)$. These probabilities represent the fractions of external nodes in the respective states. If there were n external nodes prior to insertion, there would be $n+1$ after insertion. Upon inserting into a state one external node the parent becomes full, and there are three new state two external nodes. A loss of two state one external nodes (two times $p(n)$) and a gain of three state two external nodes (three times $p(n)$). If an insertion occurs in a state two external node, the parent was full and subsequently splits. This results in the loss of three state two external nodes (three times the probability of a state two external node, $q(n)$) and the gain of four state two external nodes (four times $q(n)$).

$$(n + 1)p(n + 1) = np(n) - 2p(n) + 4q(n) \tag{2.3}$$

$$(n + 1)q(n + 1) = nq(n) - 3q(n) + 3p(n) \tag{2.4}$$

These are the difference equations for the level one analysis with the probability of inserting into a state one node, $p(n)$, and the probability of inserting into a state two node is $q(n)$. By dividing through by $(n + 1)$ the difference equations become:

$$p(n + 1) = \frac{(n - 2)}{n + 1} p(n) + \frac{4}{n + 1} q(n) \tag{2.5}$$

$$q(n + 1) = \frac{(n - 3)}{n + 1} q(n) + \frac{3}{n + 1} p(n) \tag{2.6}$$

The matrix form is given by:

$$(p(n + 1), q(n+1)) = (p(n), q(n))\begin{pmatrix} \frac{n - 2}{n + 1} & \frac{3}{n + 1} \\ \frac{4}{n + 1} & \frac{n - 3}{n + 1} \end{pmatrix} \tag{2.7}$$

From Mitchell (5, page 14), the general form of the equation is:

$$S_{n+1} = S_n*T(n) \tag{2.8}$$

where S_n and S_{n+1} are the probability vectors and T(n) is the transition matrix.

From Van Doren (6) T(n) "represents a stochastic transition matrix for a two state nonhomogeneous Markov chain" and an asymptotic solution to Equation (2.7) may be determined "which is independent of the initial frequency or probability." The limit of (p(n), q(n)) as n approaches infinity is (p,q) where p + q = 1 and (p,q) must satisfy Equations (2.5) and (2.6) where the dependence on n is deleted (6). In matrix form the limiting probability vector satisfies

$$S = S*T(n) \tag{2.9}$$

for any n. In other words the limiting probability vector is the left eigenvector corresponding to eigenvalue 1.0 and where the sum of the components is 1.0 (1.0 is necessarily an eigenvalue because the row sums for the transition matrix are 1.0).

From Mitchell (5, page 18) by solving Equations (2.5) and (2.6) algebraically, the result is:

$$((n - 2) \cdot p) + 4 \cdot q = p \cdot (n + 1)$$

$$(3 \cdot p) + ((n - 3) \cdot q) = q \cdot (n + 1). \tag{2.10}$$

By solving algebraically the result is:

$$4 \cdot q = 3 \cdot p, \tag{2.11}$$

and from p + q = 1:

$$q = 1 - p. \tag{2.12}$$

Therefore, the values for p and q are:

$$p = 4/7 \qquad q = 3/7. \tag{2.13}$$

Second Level Analysis

To expand the analysis to the second level it is necessary to expand the difference equations for the seven state second level mode. From Van Doren (6) this analysis is based upon establishing the difference equations and determining the common left eigenvector of the sequence of transition matrices. These equations (6)

$$(n + 1) * p_1(n) = n * p_1(n - 1) - 4 * p_1(n - 1) + (24/7) * p_5(n - 1 + 3 * p_6(n - 1) + 8/3 * p_q(n - 1)$$

$$(n + 1) * p_2(n) = n * p_2(n - 1) - 5p_2(n - 1) + 5 * p_1(n - 1) + (15/7) * p_6(n - 1) + (10/3) * p_7(n - 1)$$

$$(n + 1) * p_3(n) = n * p_3(n - 1) - 6 * p_3(n - 1) + (12/5) * p_2(n - 1) + 4 * p_7(n - 1)$$

$$(n + 1) * p_4(n) = n * p_4(n - 1) - 6 * p_4(n - 1) + (18/5) * p_2(n - 1)$$

$$(n + 1) * p_5(n) = n * p_5(n - 1) - 7 * p_5(n - 1) + 7 * (p_3(n - 1)) + p_4(n - 1)$$

$$(n + 1) * p_6(n) = n * p_6(n - 1) - 8 * p_6(n - 1) + (32/7) * p_5(n - 1)$$

$$(n + 1) * p_7(n) = n * p_7(n - 1) - 9 * p_7(n - 1) + (1/4) * p_6(n - 1). \qquad (2.14)$$

represent state transitions with the appropriate probabilities proportioned to reflect the fraction of external nodes which would cause a transition to this state if there is more than one transition state possible. The probabilities for the second level (6) were computed algebraically. From Van Doren (6) in Figure 10

State	Probability
1	1656/7991
2	1980/7991
3	5472/(7*7991)
4	7128/(7*7991)
5	1575/7991
6	800/7991
7	180/7991

Figure 10. Second Level Asymptotic State Probabilities

the asymptotic state probabilities for the second level are computed using the seven state second level model.

Application of Analysis Technique to Third Level

To determine the asymptotic state probabilities at the third level this method of analysis used at the first and second levels is applied to the third level. The difference equations are established for the various states, the transition matrix is established by mapping the source states of the transition to the largest states of the transitions, and the solutions are determined. For the third level analysis the asymptotic solution is found by the iterative technique, the Power

Rule (7, page 289). By choosing an initial value for the external mode count a transition matrix can be found, which is unique to that external mode count. An initial approximation is then given to the probabilities which correspond to the eigenvalue one. Then by multiplying the probabilities times the transition matrix, a new approximation is determined for the probabilities. The Power Rule states that given a sufficient number of iterations, convergence is reached. This corresponds to the stable state mentioned by Mitchell in the level one analysis. Aside from considerations given to the magnitude of the number of states the analysis is based on the first and second levels. If the asymptotic fraction of external nodes can be determined for each class, then a measure of the behavior at the asymptotic level can be observed from the utilization, the splitting probabilities, and a frequency analysis.

CHAPTER III

DETERMINATION OF THIRD LEVEL PROBABILITIES

The Relationship Between Third Level Analysis and Second Level Behavior

The third level states are defined in terms of combinations of second level states. When an insertion occurs into a middle, left, or right branch of a third level descendent, the transitions reflect the behavior of an insertion into a second level branch. Since the behavior of second level nodes upon insertion is already known, the insertions into third level states reflect merely a combination of second level actions proportioned over the fraction of external nodes which are in the given second level state. The complexity of the analysis is reduced by mapping the third level states into second level branches, performing the transitions at the second level, and then mapping the results back into the third level.

Third Level Analysis Concepts

The third level analysis (6) is based on the first and second levels. A given third level state can be decomposed into second level branches. From the preceding section on the third level the mirror images are considered and this redundancy is avoided. The third level is analyzed by setting up the difference equations and by iterating with the Power Rule. The results of the third level analysis are used for further analysis.

The establishment of the difference equations for the 224 states requires analysis of each of the states to reflect the changes which occur upon insertion. When an insertion occurs into a state one third level node, in Figure 11, there is the creation of a state two third level node.

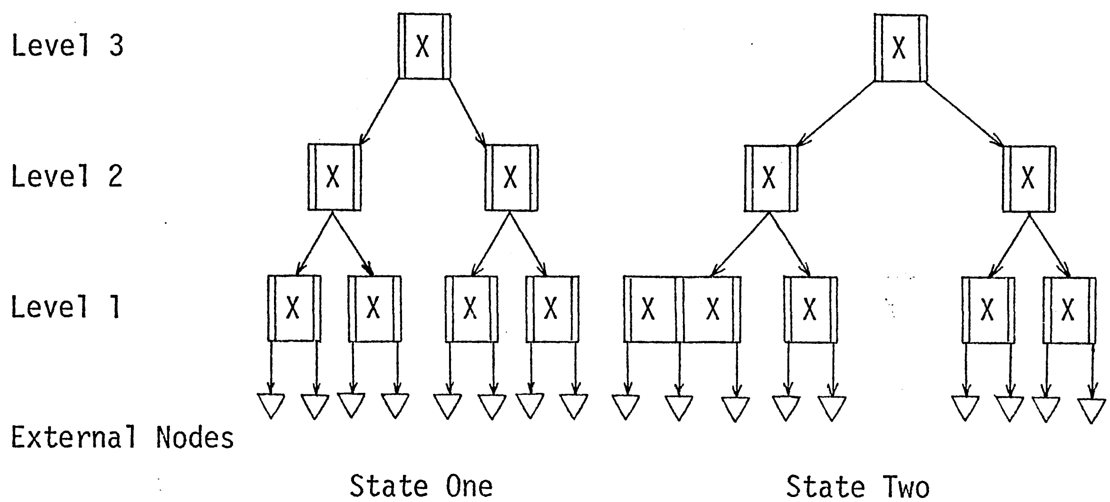


Figure 11. The Transition From State One to State Two

The full bottom level node is mapped to the most left branch regardless of the point of insertion. Hence, in the transition matrix there would be two entries. There would be an entry for the number of nodes lost to state one (eight) in the difference equation for the state one third level probability and an entry in the state two third level node difference equation to reflect the gain of nine external nodes upon insertion into state one nodes. When the insertion occurs in a third level state having two key bottom level nodes and one key bottom level node, there are several possibilities. If the insertion into the two key bottom level node propagates a key to a two key second level node, it would force the creation of new third level nodes. The result would vary according to which bottom level node is the target of insertion as to which level three state is formed. The probabilities in the target states are proportioned as to the fraction of external nodes which result in the transition to the given state times the number of external nodes created by the insertion. The first 28 states are analyzed initially as to source and target states after insertion for an insertion into the left branch and the right branch. Then the remaining 196 states are analyzed for insertions into the middle, left, and right branches and proportioned as to the percentage of external nodes residing in the second level branch.

After establishing the difference equations the values are initialized to start iterating by the Power Rule. The Power Rule (7, page 289) states that for the sequence of vectors S(k) and S(k+1) where k is the number of multiplications and for some matrix A related such that

$$S(k) = S(k + 1) * A \tag{3.1}$$

there exist values for the vectors such that

$$S(k) = S(k + 1) \quad \text{as } k \to \infty. \tag{3.2}$$

This is the eigenvector which is the asymptotic solution. Then the transition matrix is mapped with the values of the same state transitions. The same state transition reflects the transition to a given state from the previous value of that state for the previous external node count.

After establishing the initial values for the external node count and the probabilities the solution is obtained by iterating. The Power Rule does not provide the theoretical basis for the existence of a solution. The Power Rule does provide a practical means for determining the asymptotic solution when there are a large number of states. The convergence values are determined by taking the absolute values of the difference between the present probabilities and the previous values of probabilities. This is done until a limit of iterations has been done or until the convergence values are low enough to indicate asymptotic convergence. After each iteration the values are analyzed to determine the convergence values.

When the convergence values have been computed, the probabilities serve as the asymptotic state probabilities. Each value in the probabilities represents the probability of an insertion occurring in that third level tree. The 224 probabilities sum to one and represent the left eigenvector corresponding to the eigenvalue of one. These probabilities reflect the behavior of the third level at the asymptotic state for the 224 state classification mode. The data serves as input to determine the activity of the trees when a stable level is reached. This data serves as input to further analysis of the asymptotic state of third level minimum order B-trees.

Data Structures

The data structures reduce the intricacy which is introduced with analysis of this nature. The second level transitions are known prior to this analysis; hence, much of the second level results is integrated into the data structures of the third level analysis. The analysis done on the third level asymptotic state probabilities represents an extension of work done by Van Doren (6). The analysis represents a refinement to include a seven state second level model, whereas the analysis by Van Doren was with a six state second level model. The six state model does not allow the frequency analysis done by combining states three and four. States three and four have six external nodes but a state three second level node has two keys and a state four second level node has one key. In the seven state model, as was mentioned previously, there are 224 states. The task of setting up the difference equations to correspond to the mapping of the source and target states of insertion suggests a more automated approach to the establishment of the difference equations. There are a number of data structures used in doing this.

Since the middle branch value represents a mapping of segments of 28 states, the data structure LRCMB of left and right combinations can be used to uniquely determine a given thrid level state. The larger branch is on the left with 28 pairs of left-right branches. The left-right combinations in Figure 12 correspond to the left and right branches in the first 28 states.

```
LRCMB--LEFT-RIGHT COMBINATIONS OF TWO-LEVEL STATES
   DCL LRCMB(28,2) FIXED BINARY STATIC INITIAL
(1,1, 2,1, 2,2, 3,1, 3,2, 3,3, 4,1, 4,2, 4,3, 4,4,
 5,1, 5,2, 5,3, 5,4, 5,5, 6,1, 6,2, 6,3, 6,4, 6,5,
 6,6, 7,1, 7,2, 7,3, 7,4, 7,5, 7,6, 7,7);
```

Figure 12. Left-Right Combinations of Two Level Branches

The data structure TL-TRANS in Figure 13 contains second level transition pairs. It is composed of two data aggregates, TECNT and TSPRS. They are mapped together with one value of TECNT corresponding to a pair of values in TSPRS. By specifying the ith row and jth column of TECNT, the corresponding value of the transition pair ith row and jth column can be referenced. In a state one, second level tree there are four external nodes, and insertion into any one of these results in the transition to a state two second level tree. TECNT, which reflects the number of external node targets in this state transition, has the entry, four, and since the transition results in a state two the entry in row one and column is the pair two, zero. The zero implies no split at the second level. The three column entries for each row corresponds to the three possible bottom level nodes possible descending from a second level node. In a state two second level tree there are two types of bottom level nodes. One type has two keys placed on the left to avoid mirror image redundancy, and the other has one key. The two key bottom level node in state two has three external nodes, hence the entry for now two column one in TECNT is three. Insertion in this node forces a bottom level split and a state four second level tree, hence the pair for row two column one of TSPRS is four, zero. This reflects a transition to state four and no split at the second level. The second bottom level node has two external node targets; therefore, the entry in row two column two of TECNT is two. Insertion here results in the transition to a state three second level tree with no split. The entry in row two column two of TSPRS is the pair three, zero. In states three and four there are six external nodes, and each results in a transition to state five. For state five there are three bottom level nodes: one

```
TL-TRANS--TWO-LEVEL STATE TRANSITION INFORMATION
   TECNT--* OF EXTERNAL NODE TARGETS IN TWO-LEVEL STATE TRANSITION
   TSPRS--TRANSITION PAIRS(ZERO FOR RIGHT MEMBER OF PAIR MEANS NO
          SPLIT AT SECOND LEVEL)
          MAXIMUM OF THREE PAIRS FOR EACH TWO-LEVEL STATE

DCL 1 TL_TRANS STATIC,
      2 TECNT(7,3) FLOAT(16) INITIAL(
        4.0E0, 0.0E0, 0.0E0,
        3.0E0, 2.0E0, 0.0E0,
        6.0E0, 0.0E0, 0.0E0,
        6.0E0, 0.0E0, 0.0E0,
        3.0E0, 4.0E0, 0.0E0,
        3.0E0, 3.0E0, 2.0E0,
        (3)3.0E0),
      2 TSPRS(7,3,2) FIXED BINARY INITIAL(
        2,0, 0,0, 0,0,
        4,0, 3,0, 0,0,
        5,0, 0,0, 0,0,
        5,0, 0,0, 0,0,
        1,1, 6,0, 0,0,
        1,2, 2,1, 7,0,
        2,2, 1,3, 3,1);
```

Figure 13. Two Level State Transition Information

descending from the left branch with two keys, and the other two have one key each. Insertion into the full left branch results in a split, a key being propagated, and a second level split. The entry for row five column one of TECNT is three corresponding to the number of external nodes which result in the transition to two, state one second level nodes. The entry for row five, column one of TSPRS is the pair one, one. Insertion into any of the other four external nodes results in the transition to a state six second level node. Insertion into the left branch of a state six second level node results in the transition to state one and state two second level nodes. There are three external nodes which cause this transition, hence the entry for row six column one in TECNT is three. The entry in TSPRS for row six column one is the pair one, two. Insertion into the middle branch of a state six second level node results in the transition into a state two and a state one second level nodes. There are three external nodes which cause this transition, hence the entry for row six column two of TECNT is three. The entry in row six column two of TSPRS is the pair two, one. There are two external nodes descending from the right branch of a state six second level node. Insertion in the right branch results in the transition to a state seven second level node. Insertion into the left branch of a state seven second level node results in the transition to two state two second level trees. Insertion into the middle branch results in the transition to a state one and a state three second level trees, respectively. Insertion into the right branch of a state seven external node results in the transition to a state three and a state one second level trees.

Along with this data structure is the array ETAB (Figure 14), which is the number of external nodes descending from the second level states. It initializes to the external node counts for state zero to state seven. The zero entry is for a null middle branch. The entries for states three and four represent the two possible types of second level states with six external nodes.

```
DCL ETAB(0:7) FIXED BINARY STATIC INITIAL(0,4,5,6,6,7,8,9);
```

Figure 14. External Node Counts

The data structure TRANS-MATRIX in Figure 15 has the data elements LINK, STATE, and PROB. This is the state transition model in the form of difference equations which are maintained in list form. The first 224 positions are the list headers for the states. The list heads record the same state transitions.

```
TRANS-MATRIX--STATE TRANSITION MODEL IN THE FORM OF DIFFERENCE
              EQUATIONS ARE MAINTAINED (IN LIST FORM).  THE
              FIRST 224 POSITIONS ARE USED AS LIST HEADERS FOR
              THE 224 EQUATIONS.  LIST HEADS ALSO RECORD THE
              SAME STATE TRANSITIONS.
DCL 1 TRANS-MATRIX(1500) STATIC,
      2 LINK FIXED BINARY,
      2 STATE FIXED BINARY,
      2 PROB FLOAT(16),
```

Figure 15. The Transition Matrix

The last data structure is the P vector which contains the state probabilities. It is the left eigenvector of the state transition matrix corresponding to the eigenvalue one.

The Procedure to Establish the Asymptotic State Probabilities

The following section describes the actual flow of control used in the program to determine the asymptotic state probabilities at the third level. The general approach is from Van Doren (6) with a modification for a seven state model. The flow diagrams represent in general terms the actual procedures invoked. The purpose of this section is to illustrate the approach used in establishing these probabilities.

The first procedure controls the general flow of the program to establish the asymptotic state probabilities. The program is organized into modules (procedures) each with a distinct function. The concepts of modular programming are emphasized through the use of a small driver to manage the flow of control. The driver program in Figure 16 initializes the link fields in the transition matrix. Then it calls the routines GENT1 and GENT2 to establish the difference equations for the third level. The ADJUST procedure sets up the same state transition values. It also establishes the external node count a low value to allow for asymptotic convergence. Then control passes back to the driver program which calls the ECHECK procedure. The ECHECK PROCEDURE determines that the probabilities associated with the transition from a given state sum to one. Then control returns to the driver program which calls the SOLVE procedure which results in the asymptotic probabilities. The results are then used as input for further analysis.

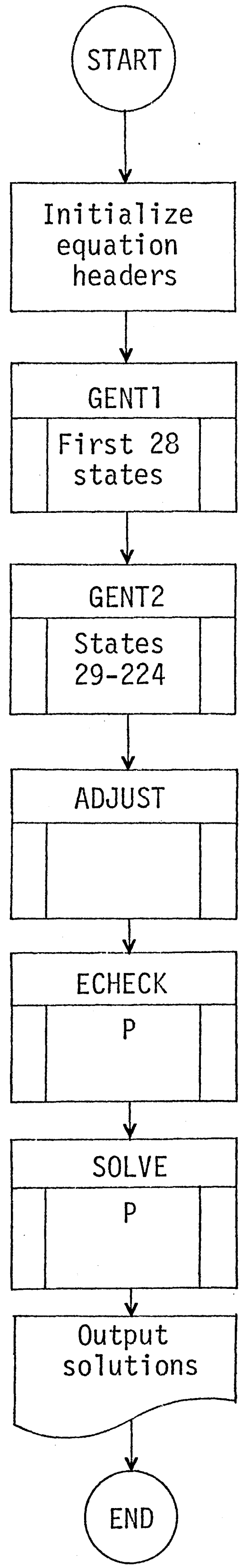


Figure 16. Driver Program

The GENT1 procedure in Figure 17 generates the transitions for the first 28 states which have a null middle branch. The driver program calls this routine and passes one of the first 28 states. The probability of having a particular state is proportioned as a fraction of the external nodes which result in a transition to the particular state. This is done for an insertion into the left and right branches. The resultant transition is placed in the transition matrix. Control returns to the driver program.

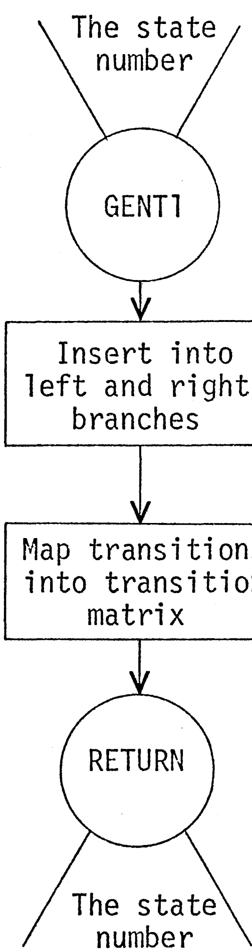


Figure 17. One Key Transitions

The driver program calls the routine in Figure 18 to map the transitions for the two key third level states. GENT2 performs insertions into any of the existing bottom level nodes descending from a second level branch. The resultant tree is mapped to the new third level state. Then the probability is proportioned to reflect the percentage of external nodes which cause this transition. Then this value is mapped into the transition matrix. This is done for each bottom level node and the process is repeated for the right and middle branches, and control reverts to the driver program.

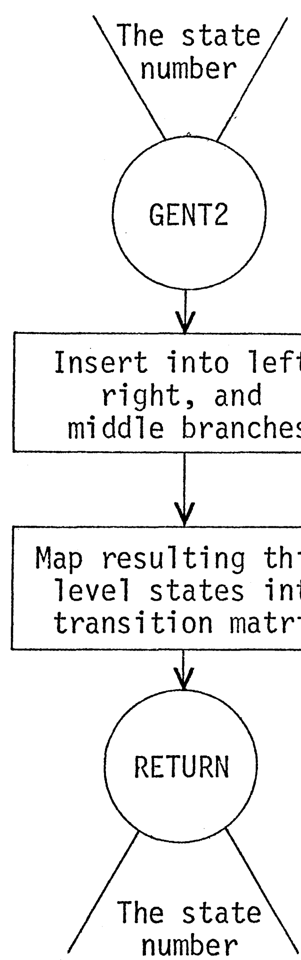


Figure 18. Two Key Transitions

The driver program calls the ADJUST procedure in Figure 19 to set up the same state transition and also sets the external node count at a low value, 30, to allow for convergence by the Power Rule. After processing the 224 states, control returns to the driver program.

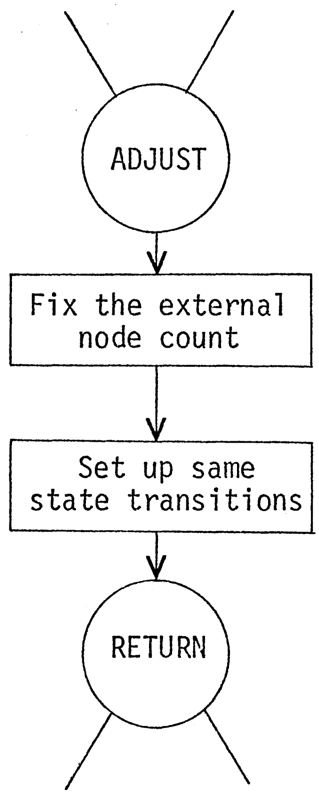


Figure 19. The Adjust Routine

Then the driver program calls the ECHECK procedure in Figure 20 which sums the probabilities associated with invocation of a particular state. The sum of the probabilities associated with invoking a given state should sum to one. This routine essentially ascertains that the difference equations are set up correctly in the sense that they are probabilistically consistent in Figure 21. Control resumes to the driver program.

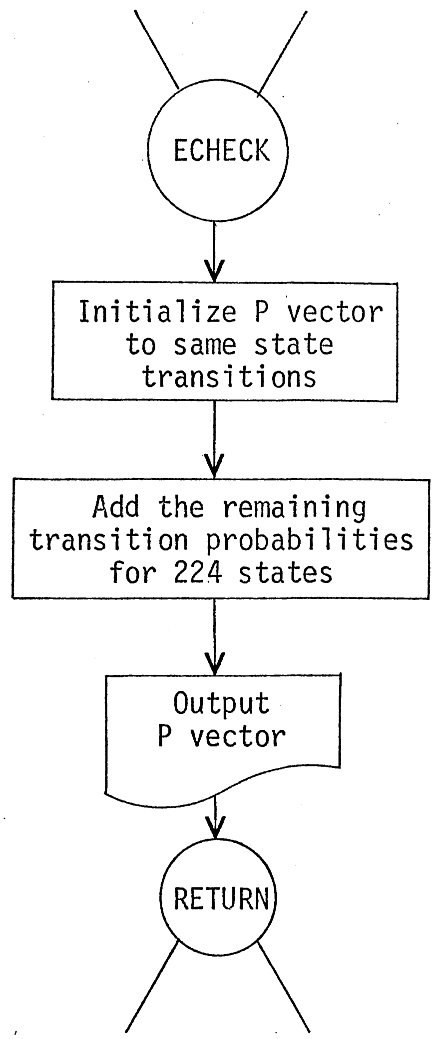


Figure 20. The Error Check Routine

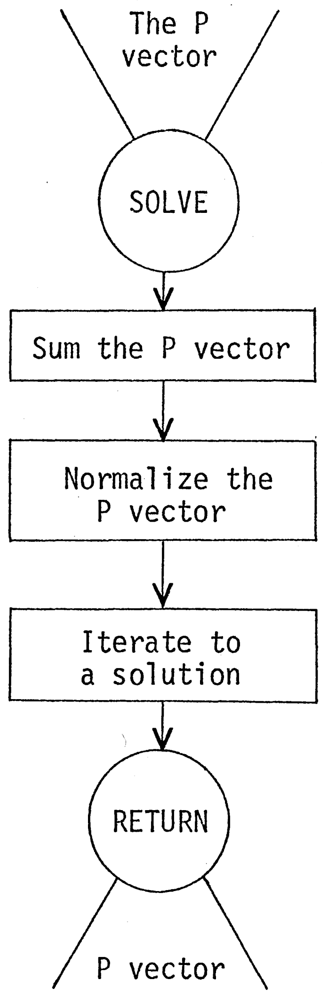


Figure 21. The Routine to Obtain a Solution

Subsequently the driver program invokes the SOLVE routine which iterates to a solution by using the Power Rule. Control returns to the main program which outputs the results for subsequent analysis.

CHAPTER IV

ANALYSIS OF THE THIRD LEVEL PROBABILITIES

There are various types of analysis that can be performed upon the asymptotic probabilities. These probabilities reflect a relative frequency of occurrence for each of the 224 external node states. There are some important relationships to be determined about the behavior of 2-3 trees. The probability of a split at a given level can be determined given that the previous level has propagated a key forcing the split. The probability of a split is determined for all three levels. The utilization at the various levels is important to determine the efficiency of the asymptotic or third level. Finally, the frequency of nodes is determined which descend from one and two key nodes at the second level, and one and two key nodes at the first level. The analysis of the asymptotic behavior of 3-2 trees is based on the determination of the utilization of the available key positions for the nodes, the probability of a split for the three levels, and the frequency of nodes descending from one and two key parents at the various levels.

The Utilization

The utilization is the number of keys in the nodes divided by the number of key positions available. The utilization in the subsequent section on the results. From Van Doren (6) the utilization can also be computed from the probability of splitting by the relationship

$$u(m) = (1 - ps(m))/(2 * ps(m))$$

where u(m) is the utilization for level m, and ps(m) is the conditional probability of splitting at level m. The asymptotic probabilities for the external node states serve as the input for a measure of the frequencies of occurrence for the various states. Let p(i) be the asymptotic probability that an external node is in state i and let n(i) be the number of external nodes which necessarily occur together in state i. For every n(i) such external nodes there will be one three level 2-3 subtree. That is to say

$$\frac{p(i)}{n(i)} \tag{4.2}$$

represents a relative frequency of occurrence of 2-3 subtrees whose external nodes are in state i. The first 28 states represent subtrees with a one key node (null middle branch) at level three. Letting S1 be the relative frequency of one key nodes at level three and using Equation (4.2)

$$\sum_{i=1}^{28} \frac{p(i)}{n(i)} = S1 \tag{4.3}$$

The relative frequency, S2, of two key nodes for the third level nodes is

$$\sum_{i=29}^{224} \frac{p(i)}{n(i)} \tag{4.4}$$

The utilization is then computed as

$$\frac{S1 + 2 * S2}{2 * (S1 + S2)} \; . \tag{4.5}$$

The numerator in Equation (4.5) represents the relative number of keys at level three and the denominator represents the relative number of key positions, two for each node.

The probabilities of occurrence for the third level serve as input to the computation of the utilization at level two because they measure the relative frequencies of occurrence. Since each third level state represents a unique combination of left, right, and middle branches, the third level states can be mapped to the second level. The probabilities reflect the frequency of occurrence of that combination of second level nodes. The probability of occurrence of a given third level state is proportioned to the one or two key second branches by the fraction of the external nodes which reside in the specific branch. Each of the third level states is decomposed into left, middle and right branches, and their relative frequencies are added to the one key sum and the two key sum. The analysis concludes for the second level by using the relationship in Equation (4.5) to determine the utilization at level two.

The first level is analyzed by taking a given third level state and decomposing it into second level states. The second level states are decomposed into the number of one key and two key nodes. The number of one key nodes is multiplied times the probability of this third level classification occurring divided by the external node count. The sum of two key nodes is done in a similar manner. After determining the relative sums for one key and two key nodes at the first level for all 224 third level states Equation (4.5) is applied to the sums.

The Probability of a Split

The asymptotic state probabilities serve as input to the computation of the probability of a split. The results of the computations of the probabilities of splitting are presented in a subsequent section.

The probability of a split at a given level implies a key being propagated into this level from a lower level, and that the key is propagated into a full node at the given level. The probability of a split at the third level implies a split at the second level from a key propagated from a level one split since all insertions in a B-tree occur at the first level. The analysis of the probability of a split at level three considers only the full level three nodes and the possibility of a key being propagated to them. The probability of a level three split is found from the fraction of external nodes causing a level three split times the probability of this state occurring. Let F1 be the number of external nodes which lead to a level three split, m be the number of external nodes, and p be the probability of state i occurring. The probability of a three level split is

$$\sum_{i=29}^{224} \frac{F1(i)}{n(i)} * p(i) \qquad (4.6)$$

The probability of a split at the second level infers the decomposition of each third level state into its middle, left, and right branches. Then the second level states that are full are separated from the one key second level nodes. The full second level nodes are decomposed into the number of external nodes which descend from full bottom level nodes. The fraction of the external nodes from a level three state which cause a level two split, F2, is multiplied times the probability of that state occurring. The equation is

$$\sum_{i=1}^{224} \frac{F2(i)}{n(i)} * p(i) \qquad (4.7)$$

for state "i." The sum represents the probability of a split at level two.

The level one probability of a split involves the decomposition of the level three node into level one state one and two nodes. The fraction of the external nodes from a level three node which descends from a two key bottom level node, F3, is multiplied times the probability of

$$\sum_{i=1}^{224} \frac{F3(i)}{n(i)} * p(i) \tag{4.8}$$

this level three state occurring. The sum of these probabilities represents the probability of a split at level three.

The Frequency Analysis

The frequency of keys descending from a given one or two key parent, grandparent is computed from the asymptotic state probabilities. The frequencies which are given by position in Figure 22 are computed by initially decomposing a given third level state into middle, left, and right branches. The results of the frequency analysis are presented in a subsequent section. To determine the second level frequency of a given node descending from a one or two key third level parent, the probability of a third level parent is divided by the external node count. Each type of second level node is summed with the subsequent occurrences of this type. The first level nodes are analyzed as to the classifications of one or two key parents and one or two key grandparents. Then the first level nodes descending from a given second level parent and grandparent are computed by multiplying the precent of external nodes in this state times the probability of this state occurring. These are the relative frequencies for each type of first level node and the third level state serves as the data to compute the frequencies. The results of the frequency analysis are presented in a form which indicates by the relative position the frequency which appears in the tree.

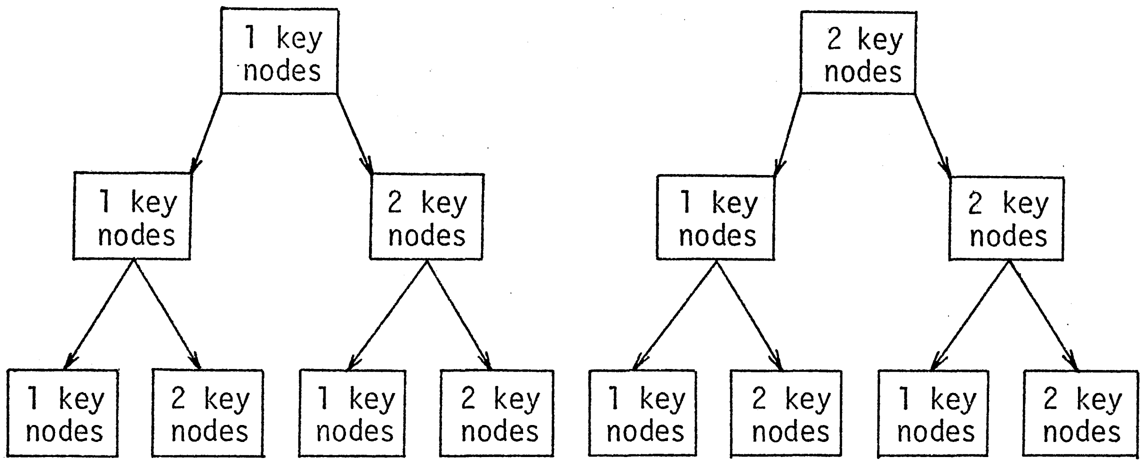


Figure 22. Frequency Representations

CHAPTER V

THE RESULTS OF THE ANALYSIS

Once the asymptotic values of occurrence have been computed for the 224 states, analysis is made of the three levels of nodes. The probability of a split at the various levels is determined by knowing the behavior of the various states. The probability of a split represents the fraction of the occurrences of a key being propagated from the level below the given level into a full node. The probability of a split for the various levels is the sum of the products of the probability of occurrence of a given two key state under random insertion times the percentage of external nodes which would cause a key to be propagated. The utilization is the total number of keys divided by two times the number of nodes. This is a measure of occupancy which reflects the percentage of key slots that are occupied. The data structures are used to avoid much of the redundancy of data concerning the states. The logic is the facilitation of the relationships with the use of the asymptotic probabilities as the input for the computation for the three levels.

The first data structure used in the analysis is the probability vector, PROB, containing the asymptotic probabilities of occurrence. The next data structure in Figure 23 is ETAB, which contains the number of external nodes descending from a level two node. The values correspond to state zero, state one, and through state seven. State zero is for a null middle branch. The next data structure, EXNDS, in Figure 24 is initialized to the number of external nodes descending from the respective level two state which force a level one split. There is no zeroth entry.

```
DCL ETAB(0:7) FLOAT(16) STATIC INITIAL(
      0.0E0, 4.0E0, 5.0E0, 6.0E0,
      6.0E0, 7.0E0, 8.0E0, 9.0E0);
```

Figure 23. The External Node Counts

```
DCL EXNDS(7) FLOAT(16) STATIC INITIAL(
      0.0E0, 3.0E0, y.0E0, 0.0E0,
      3.0E0, 6.0E0, 9.0E0);
```

Figure 24. The External Nodes Descending from Full Nodes

The last major data structure in Figure 25 is LRCMB, which initialized to contain the possible left and right branch combinations descending from a third level node. It is dimensioned 28 by 2 and contains the left and right branches of the first 28 third level states.

```
   DCL LRCMB(28,2) FIXED BINARY STATIC INITIAL(
1,1, 2,1, 2,2, 3,1, 3,2, 3,3, 4,1, 4,2, 4,3, 4,4,
5,1, 5,2, 5,3, 5,4, 5,5, 6,1, 6,2, 6,3, 6,4, 6,5,
6,6, 7,1, 7,2, 7,3, 7,4, 7,5, 7,6, 7,7);
```

Figure 25. The Left-Right Combinations

The Procedure in the Third Level Analysis

These diagrams represent a general flow of the logic to determine the probability of a split, the utilization, and determine the frequency of nodes in a specific classification. The program is broken into modules which accomplish the analysis. The program is controlled through a driver program in Figure 26 which invokes the procedures to perform the analysis. Initially the driver program reads the probabilities and normalizes the input. Then the driver program calls the procedure to compute the utilization. This procedure computes the utilization and performs the frequency analysis. Then control returns to the main procedure which ends the analysis.

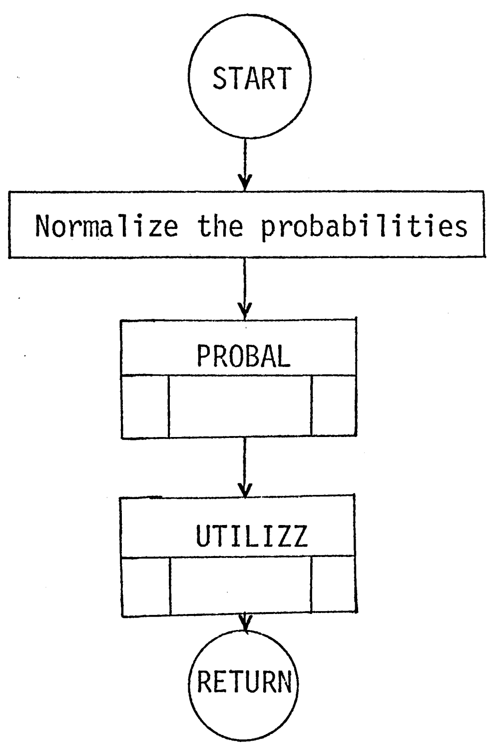


Figure 26. The Driver Program

The routine in Figure 27 computes the probabilities for splitting at the various levels. It computes the third level probability by decomposing the full third level nodes into the external nodes which cause a level two split beneath the full node. Then the fraction of the nodes which a split is multiplied times the probability of the third level state occurring. The sum of these probabilities represents the probability of a split at the third level, which is (to eight digits of accuracy)

$$.07452526.$$

Subsequently the procedure computes the probability of a split at the second level. It is done by decomposing each third level state into left, middle, and right branches. Then for the full second level nodes the external nodes are summed which force a level one split. The fraction of external nodes which cause a level two split is multiplied times the probability of the third level state occurring. The sum of these probabilities is the probability of a split at the second level, which is

$$.18207983.$$

The probability of a split at the first level is computed by decomposing each third level state into the fraction of external nodes descending from full level one nodes. Then the percent of external nodes which descend from full bottom level nodes is multiplied times the probability of that state occurring. The sum of these probabilities is the probability of a split at the first level, which is

$$.42857142.$$

Then control returns to the driver program.

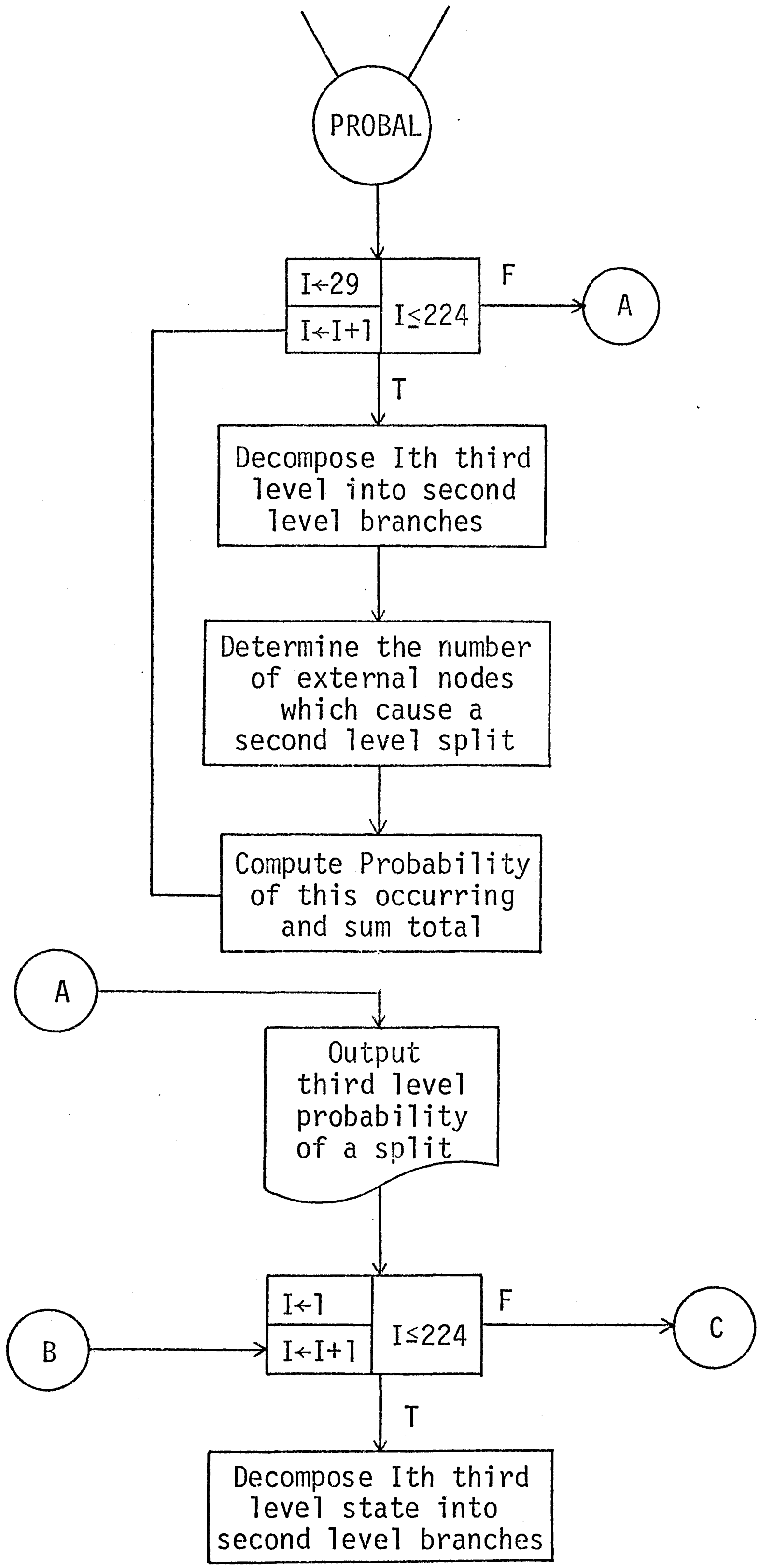


Figure 27. The Probability of a Split

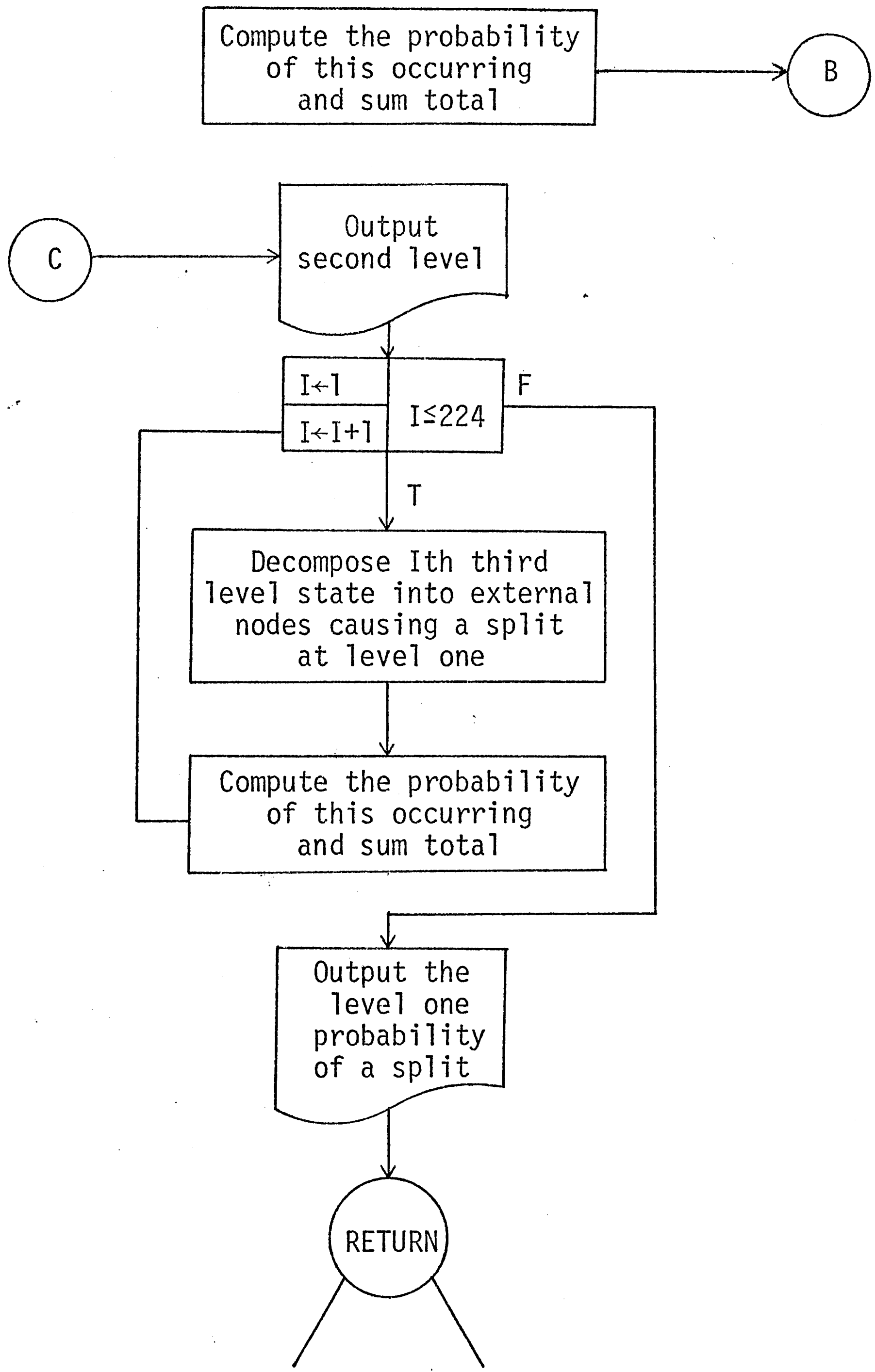


Figure 27. (Continued)

The UTILIZZ procedure in Figure 28 computes the utilization for the three levels and the frequency of nodes descending from one or two key parents and grandparents. The level three utilization is computed by determining the relative frequency of one key and two key nodes. Then these frequencies are normalized and divided by two, which is the utilization. The utilization at level three is

.67542867.

The utilization at level two is computed in a similar manner after determining the relative frequencies of one and two key nodes at the second level. The utilization at level two is

.67687776.

After determining the relative frequencies of one and two keys at the first level the utilization is computed for the first level. Then the relative frequency of nodes descending from one and two key third level nodes is computed. The relative frequency of nodes descending from one and two key nodes at the third level and one and two key nodes at the second level. The subsequent material contains an analysis of the results. Then the control returns to the main program.

Frequency Analysis

The values at the top level in Figure 29 represent the relative frequency of each type of third level node (1 or 2 key) which were determined from the value of the utilization which was known. The frequencies at the second level represent the relative frequencies of a key appearing at the second level with one or two keys descending from a one or two key parent. The frequencies at the first and second levels were normalized for each level, respectively. The frequencies at the bottom

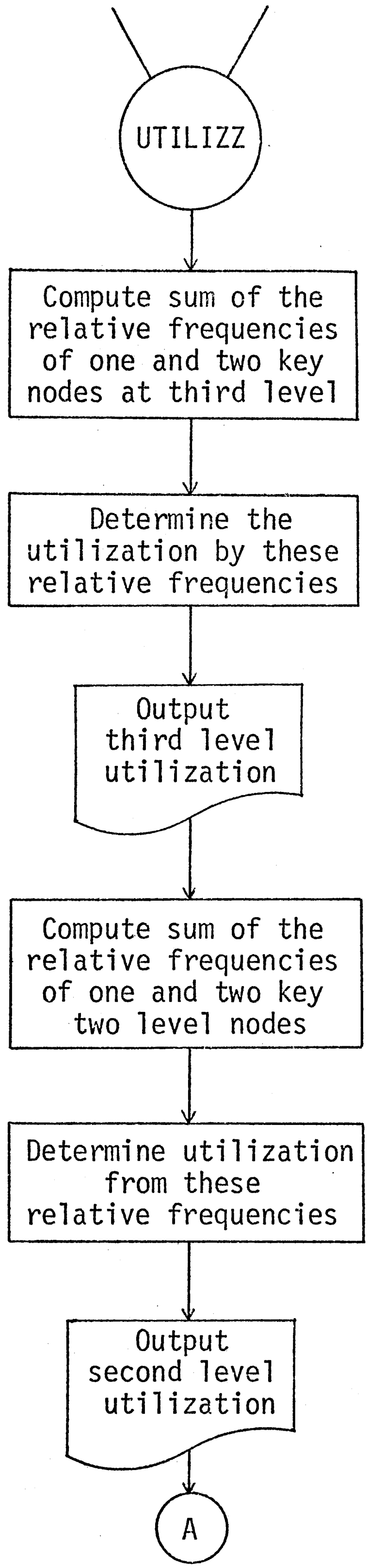


Figure 28. The Utilization and Frequency Analysis

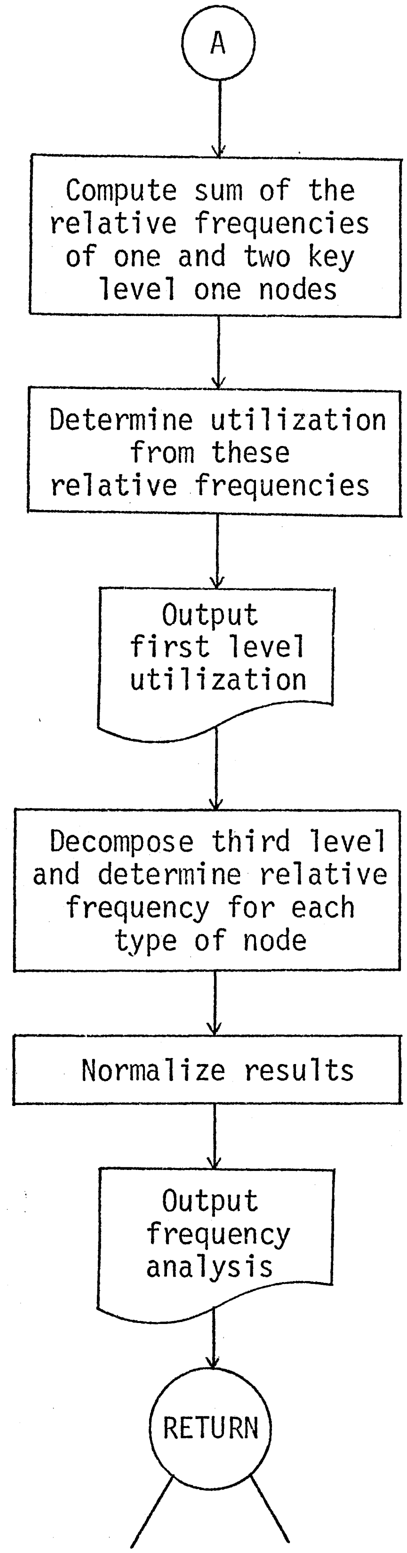


Figure 28. (Continued)

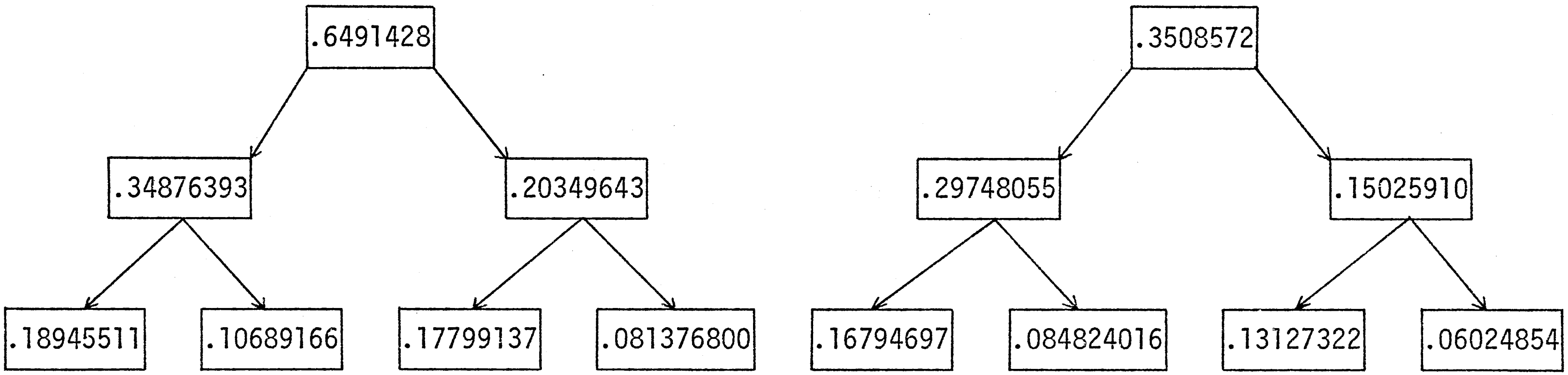


Figure 29. The Frequencies for the Various Levels in Tree Representation

level represent the relative frequencies of a key appearing at the first level descending from a one or two key parent and a cone or two key grandparent. For example, the relative frequency of nodes at the bottom level which has a one key grandparent, a two key parent, having one key is 1.7799137E-01.

To analyze how the distribution of one key and two key nodes at level i is dependent on the parent type at level i + 1, the following ratios were established. The reciprocals were included so that the results can be used for inferences from level i + 1 to level i, as well as the inferences of the relationships from level i to level i + 1.

1. The ratio of the frequency of the first level nodes with one key descending from one key third level nodes and one key second level nodes over the frequency of one key second level nodes with one key parents is

$$\frac{.18945511}{.34876393}$$

or .54321876. The reciprocal is 1.8408789.

2. The ratio of the frequency of first level nodes with one key descending from a two key third level node and a one key second level node over the frequency of a one key second level node with a two key third level parent is

$$\frac{.16794697}{.29748055}$$

or .56456454. The reciprocal is 1.7712766.

3. The ratio of the frequency of first level two key nodes descending from a third level one key node and a second level one key node over

the frequency of one key second level nodes with one key parents is

$$\frac{.10689166}{.34876393}$$

or .30648714. The reciprocal is 3.2627796.

4. The ratio of the frequency of first level two key nodes descending from a two key third level node and a one key second level node over the frequency of one key second level nodes with a two key third level parent is

$$\frac{.084824016}{.29748055}$$

or .28514138. The reciprocal is 3.5070321.

5. The ratio of the frequency of first level one key nodes descending from a one key third level node and a second level two key over the frequency of two key second level nodes having a one key third level parent is

$$\frac{.17799137}{.20349643}$$

or .87466581. The reciprocal is 1.1432938.

6. The ratio of the frequency of first level one key nodes descending from a third level two key nodes and a second level two key node over the frequency of two key second level nodes having a two key third level parent is

$$\frac{.13127322}{.15025910}$$

or .87364572. The reciprocal is 1.1446287.

7. The ratio of the frequency of first level nodes with two keys descending from third level one key nodes and second level two key nodes

over the frequency of two key second level nodes having a one key third level parent is

$$\frac{.081376800}{.20349643}$$

or .39989301. The reciprocal is 2.5006688.

8. The ratio of the frequency of a first level to key node descending from a third level two key node and a second level two key node over the frequency of a two key second level node having a two key parent is

$$\frac{.060240854}{.15025910}$$

or .40091318. The reciprocal is 2.4943056.

Furthermore, since the third level states have no bearing on the analysis of the first level frequencies given a second level of frequency, the number of states can be reduced from eight to four with these values for the frequencies shown in Figure 30. Subsequently, the following ratios are established:

1. The ratio of one key nodes with a one key parent relative frequency over the relative frequency of second level one key nodes is

$$\frac{.35740208}{.64624448}$$

or .55304469. The reciprocal is 1.8081721.

2. The ratio of two key level one nodes with one key parents relative frequency over the relative frequency of second level nodes is

$$\frac{.19171567}{.64624448}$$

or .29666121. The reciprocal is 2.9666121.

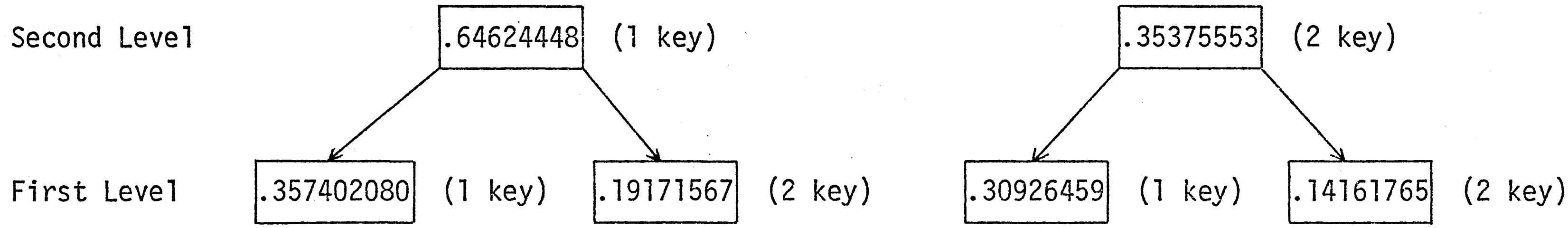


Figure 30. Second Level to First Level Frequencies

3. The ratio of one key level one nodes relative frequency over the relative frequency of second level nodes one key nodes is

$$\frac{.30926459}{.35375553}$$

or .8742352. The reciprocal is 1.1438604.

4. The ratio of two key level one nodes relative frequency over the relative frequency of second level two key nodes is

$$\frac{.14161765}{.35375553}$$

or .40032632. The reciprocal is 2.4979621.

To illustrate what these relationships mean, if the second level frequency for one key nodes was given, the first level frequency for one key nodes can be determined by multiplying by the reciprocal of this state, 1.8081721, by the given second level frequency. Also, the frequency of two key level one nodes can be computed by multiplying the given frequency by the reciprocal for that state, 2.9666121. Therefore, if given the frequency of one key or two key nodes at the second level, predictions can be made for the frequency of one key and two keys at the first level.

The following ratios have been established from the level one frequency over the level three frequencies for one and two keys.

I. For One Key at the Third Level

A. With One Key at the Second Level

1. Having one key at the first level, the ratio of the level one frequency to the level three frequency is

$$\frac{.18945511}{.6491428}$$

or .29185428. The reciprocal is 3.4263674.

2. Having two keys at the first level, the ratio of the level one frequency to the level three frequency is

$$\frac{.10689166}{.6491428}$$

or .16466586. The reciprocal is 6.0729042.

B. With Two Keys at the Second Level

1. Having one key, the ratio of the level one frequency to the level three frequency is

$$\frac{.17799137}{.6491428}$$

or .27419447. The reciprocal is 3.6470465.

2. Having two keys at the first level, the ratio of the level one frequency to the level three frequency is

$$\frac{.081376800}{.6491428}$$

or .12536039. The reciprocal is 7.9770013.

II. For Two Keys at the Second Level

A. With One Key at the Second Level

1. Having one key, the ratio of the level one frequency to the level three frequency is

$$\frac{.16794697}{.3508512}$$

or .47867613. The reciprocal is 2.0890951.

2. Having two keys, the ratio of the level one frequency to the level three frequency is

$$\frac{.084824016}{.3508572}$$

or .24176222. The reciprocal is 4.1362955.

B. With Two Keys at the Second Level

1. Having one key at the first level, the ratio of the level one frequency to the level three frequency is

$$\frac{.13127322}{.3508572}$$

or .37414999. The reciprocal is 2.6727249.

2. Having two keys at the first level, the ratio of the level one frequency to the level three frequency is

$$\frac{.060240854}{.3508572}$$

or .17169621. The reciprocal is 5.8242403.

These ratios are established from the level two frequencies over the level three frequencies for one and two keys.

A. For One Key at the Third Level

1. Having one key at the second level, it is

$$\frac{.34876393}{.6491428}$$

or .53726842. The reciprocal is 1.861267,

After Normalizing .63152082.

2. Having two keys at the second level, it is

$$\frac{.20349643}{.6491428}$$

or .31348484. The reciprocal is 5.1899469,

After Normalizing .36847914.

B. For Two Keys at the Third Level

1. Having one key at the second level, it is

$$\frac{.29748055}{.3508572}$$

or .84786787. The reciprocal is 1.179429,

After Normalizing .66440519.

2. Having two keys at the second level, it is

$$\frac{.15025910}{.3508572}$$

or .42826283. The reciprocal is 2.3350146,

After Normalizing .3355948.

The following diagram (Figure 31) represents the relative frequencies in a different manner. The pair of nodes descending from a given node from the previous diagram have been summed, and each node in this diagram represents the percent in one key and two key nodes. The node to the left is a one key and to the right is a two key node. This establishes the tendency of the ratio of one key nodes to the two key nodes at 2 to 1, the percent being in one key nodes is approximately 65 percent and in two key nodes is 35 percent.

The frequencies for the third level were established by the following computation using known relations. Since the utilization for level three is known, the relative frequency for one and two key nodes at the third level can be computed from the utilization. The utilization is

$$\frac{\text{the number of keys}}{\text{2 times the number of nodes}} \tag{5.1}$$

Let the number of nodes per level be nds. The number of keys is the number of one key nodes per level plus two times the number of two key

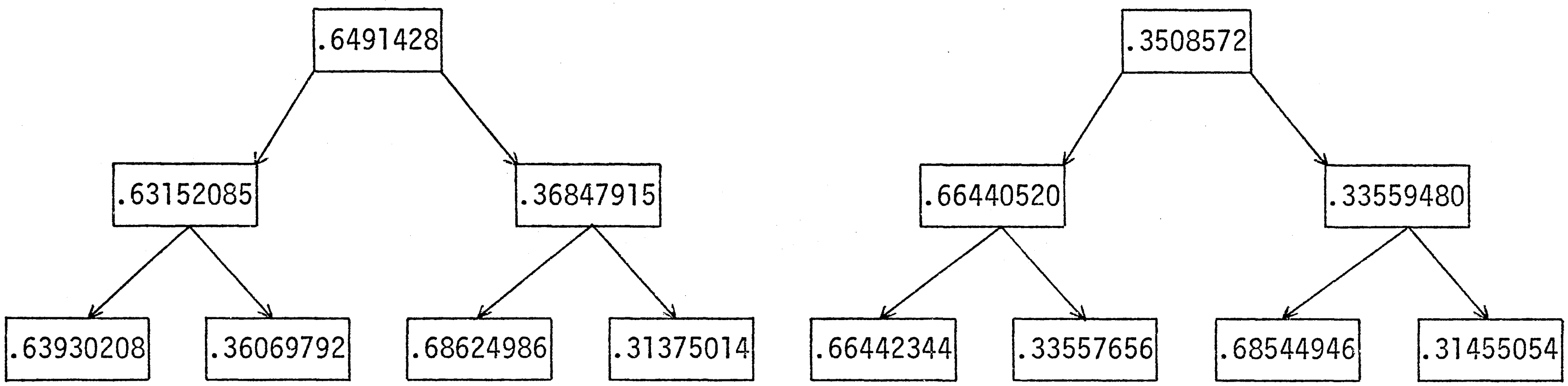


Figure 31. The Normalized Frequencies

nodes per level. Let the number of one key nodes be ONEKYNDS and the number of two key nodes be TWOKYNDS, then the utilization is equal to

$$\frac{\text{ONEKYNDS} + 2 * \text{TWOKYNDS}}{2 * \text{NDS}} . \tag{5.2}$$

Furthermore, for a given level this relation holds:

$$\frac{\text{ONEKYNDS}}{\text{NDS}} + \frac{\text{TWOKYNDS}}{\text{NDS}} = 1. \tag{5.3}$$

Hence,

$$\frac{\text{TWOKYNDS}}{\text{NDS}} = 1 - \frac{\text{ONEKYNDS}}{\text{NDS}} . \tag{5.4}$$

$$\text{Utilization} = \frac{\text{ONEKYNDS}}{2 * \text{NDS}} + \frac{(\text{TWOKYNDS})}{\text{NDS}} \tag{5.5}$$

$$\text{Utilization} = \frac{\text{ONEKYNDS}}{2 * \text{NDS}} + 1 - \frac{\text{ONEKYNDS}}{\text{NDS}}$$

$$= 1 - \frac{\text{ONEKYNDS}}{2 * \text{NDS}} . \tag{5.6}$$

The relative frequency of one key nodes at the third level is

$$\frac{\text{ONEKYNDS}}{\text{NDS}} . \tag{5.7}$$

Solving for this, knowing the utilization,

$$\frac{\text{ONEKYNDS}}{2 * \text{NDS}} = 1 - .67542867$$

$$\frac{\text{ONEKYNDS}}{2 * \text{NDS}} = .3245714$$

$$\frac{\text{ONEKYNDS}}{\text{NDS}} = .6491428$$

$$\frac{\text{TWOKYNDS}}{\text{NDS}} = 1 - \frac{\text{ONEKYNDS}}{\text{NDS}}$$

$$\frac{\text{TWOKYNDS}}{\text{NDS}} = .3508572 \tag{5.8}$$

To apply these relationships to the second level would imply use of the utilization for level two which is .67687776

$$\frac{\text{ONEKYNDS (LEVEL TWO)}}{2 * \text{NDS (LEVEL TWO)}} = 1 - .67687766$$

$$= .32312234$$

$$\frac{\text{ONEKYNDS (LEVEL TWO)}}{\text{NDS (LEVEL TWO)}} = .64624468 \tag{5.9}$$

and

$$\frac{\text{TWOKYNDS (LEVEL TWO)}}{\text{NDS (LEVEL TWO)}} = .35375538 \tag{5.10}$$

This compares with the cumulative frequencies for the second level which are .6462448, for one key, and .35375553 for two keys.

With the level one utilization $.6\overline{6}$, the node frequency is $.6\overline{6}$ for one key nodes and $.3\overline{3}$ for two key nodes from computation using the utilization. From the sums of the one key frequency and two key frequencies determined from the asymptotic probability analysis, the node frequencies for one key nodes is .66666667 and for two key nodes is .33333332 with some degree of roundoff error.

The following computation in Figure 32 is based on using the sum of the probabilities at the asymptotic state for the first 28 states to establish the relative frequency of the one key third level state and the rest of the probabilities of the remaining third level states for the two key third level relative frequency.

I. For One Key at the Third Level

A. With One Key at the Second Level

1. Having one key at the first level over the one key level three probability the ratio is

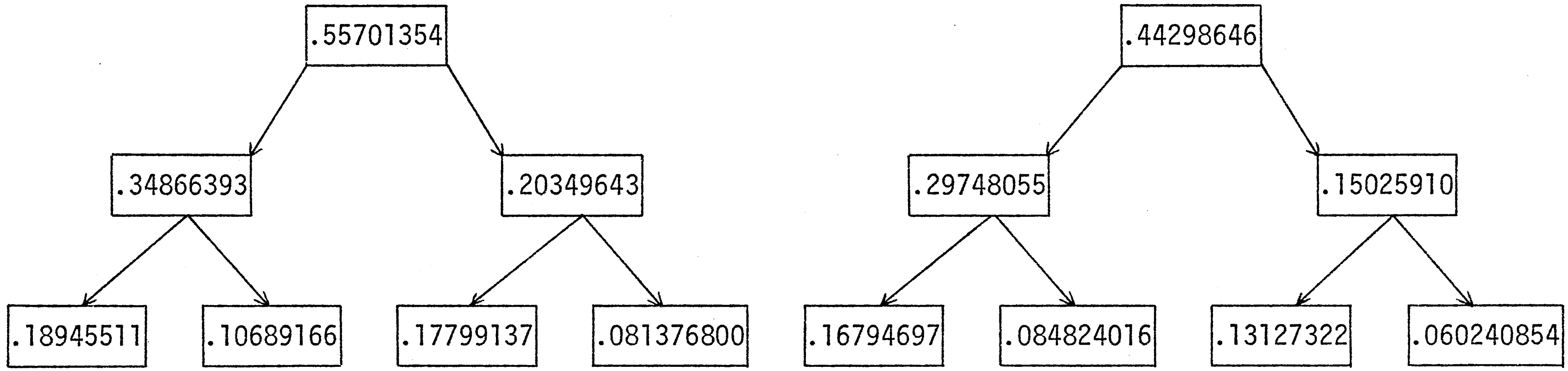


Figure 32. Frequency Analysis Using the Sum of the Probabilities

$$\frac{.18945511}{.55701354}$$

or .34012657. The reciprocal is 2.9400819,

After Normalizing .34092137.

2. Having two keys at the first level over the one key level three probability, the ratio is

$$\frac{.10689166}{.55701354}$$

or .19190136. The reciprocal is 5.2110104,

After Normalizing .19234979.

B. With Two Keys at the Second Level

1. Having one key at the first level over the one key level three probability, the ratio is

$$\frac{.17799137}{.55701354}$$

or .31954585. The reciprocal is .31294413,

After Normalizing .32029256.

2. Having two keys at the first level over the one key level three probability, the ratio is

$$\frac{.081376800}{.55701354}$$

or .14609483. The reciprocal is 6.8448691,

After Normalizing .14643622.

II. For Two Keys at the Third Level

A. With One Key at the Second Level

1. Having one key at the first level over the two key level three probability, the ratio is

$$\frac{.16794697}{.44298646}$$

or .379121438. The reciprocal is 2.6376568,

After Normalizing .37801626.

2. Having two keys at the first level over the two key level three probability, the ratio is

$$\frac{.084824016}{.44298646}$$

or .19148218. The reciprocal is 5.222418,

After Normalizing .109225.

B. With Two Keys at the Second Level

1. Having one key at the first level over the two key level three probability, the ratio is

$$\frac{.13127322}{.44298646}$$

or .29633686. The reciprocal is 3.374538,

After Normalizing .29547071.

2. Having two keys at the first level over the two key level three probability, the ratio is

$$\frac{.060240854}{.44298646}$$

or .13598802. The reciprocal is 7.3535889,

After Normalizing .13559054.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

The values in Figure 33 are the results of the computations to determine the probability of a split, the conditional probability of a split, and the utilization.

The Probability of a Split

Level 3	.077452526
Level 2	.18207983
Level 1	.42857142

The Conditional Probability of a Split

Level 3	.42537674
Level 2	.42485290
Level 1	.42857142

The Utilization

Level 3	.67542867
Level 2	.676877761
Level 1	.66666666

The conditional probability of a split is computed from the probability of splitting at each level. The conditional probability of a split is the probability of a split at level i given a random key being propagated from level i-1. The conditional probability of a split at level 1 is equal to the level 1 probability of a split. At level 2 the conditional probability of a split is the quotient of the probability of a split at level 2 divided by the probability of a split at level 1. The conditional probability of a split at level 3 is computed as the quotient of the probability of a split at level 3 divided by the level 2 probability of a split. This suggests but does not prove inferences about the behavior of each level upon propagation of a key.

With the asymptotic probabilities the third level analysis determined behavioral characteristics of 2-3 trees. For the relative frequency of having to perform a reorganization at the third level, a reorganization is necessary after ten insertions on the average or approximately for about 7.45 percent of the insertions. A second level reorganization is necessary after every five insertions or for 18.2 percent of the insertions force a second level split. A first level split occurs for 42.8 percent of the insertions. This implies the norm for maintenance at the asymptotic state.

The utilization is a measure of the efficiency for the use of the relative key space. Roughly two-thirds of the key space is occupied in each of the three levels. The frequency analysis establishes this relation by normalizing the descendents from a given node. For a given node the descendents are approximately 64 in one key nodes and 36 in two key nodes. The nodes are proportioned to the one key and two key nodes throughout the three levels analyzed.

It is hoped that this research will aid in the determination of the favorableness of the use of 2-3 trees for the storage of data.

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APPENDIX A

ASYMPTOTIC PROBABILITIES

```
$PLC TIME=05,ATR,XREF

*OPTIONS IN EFFECT*     TIME=(5,0),PAGES=30,LINES=1500,ATR,XREF,FLAGW,NOBNDRY,NOCMNTS,SORMGIN=(2,72,1),ERRORS=(3,50),
*OPTIONS IN EFFECT*     TABSIZE=21732,UDEF,SOURCE,OPLIST,NOCMPRS,HORPG,AUXIO=10000,LINECT=50,NOALIST,MCALL,MTEXT,DUMP=(
*OPTIONS IN EFFECT*     S,F,L,E,U,R),DUMPE=(S,F,L,E,U,R),DUMPT=(S,F,L,E,U,R)

 (NOSIZE,NOFLOW):                                                          PL/C-R7.1--68 05/16/77 19:45 PAGE   1

  STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

    1                         (NOSIZE,NOFLOW):
                              S2_3L3:
                              PROCEDURE OPTIONS(MAIN):
                              /*
                              LIST OF VARIABLES:
                                    PROCEDURES:
                                       ADJUST     THE ROUTINE WHICH ESTABLISHES THE SAME STATE
                                                  TRANSITIONS.
                                       ECHECK     THE ROUTINE WHICH CHECKS TO SEE THAT THE PROBABILITIES
                                                  ASSOCIATED WITH EACH GIVEN STATE SUM TO ONE.
                                       GENT1      THE ROUTINE WHICH MAPS THE TRANSITIONS FROM EACH
                                                  GIVEN STATE (THE SOURCE STATE) TO THE VARIOUS TARGET
                                                  STATES FOR EXTERNAL NODES DESCENDING FROM ONE KEY
                                                  THIRD LEVEL NODES.
                                       GENT2      THE ROUTINE WHICH MAPS THE TRANSITIONS FOR STATES
                                                  HAVING TWO KEY THIRD LEVEL NODES.
                                       ITER       THE ROUTINE WHICH DETERMINES THE NEW PROBABILITIES
                                                  FOR THE STATES FROM THE PRODUCT OF THE PREVIOUS
                                                  PROBABLITIES AND THE TRANSITION MATRIX.
                                       MAP        THE FUNCTION TO DETERMINE A THIRD LEVEL STATE FROM
                                                  GIVEN LEFT, RIGHT, AND MIDDLE BRANCHES.
                                       SOLVE      THE ROUTINE WHICH DETERMINES THE SOLUTIONS FOR THE
                                                  PROBABLITIES.
                                       S2_3L3     THE DRIVER PROGRAM.
                                       TMATX      THE ROUTINE WHICH LINKS THE TRANSITIONS INTO THE
                                                  TARGET STATES FROM THE SOURCE STATES.
                                    ARRAYS:
                                       ETAB       THE NUMBER OF EXTERNAL NODES IN A GIVEN TWO LEVEL STATE.
                                       LRCMB      THE LEFT AND RIGHT COMBINATIONS OF TWO LEVEL STATES.
                                       P          THE PROBABLITY VECTOR CORRESPONDING TO THE EIGENVALUE
                                                  ONE.
                                       PNEW       THE NEW PROBABLITIES.
                                       POLD       THE OLD PROBABLITIES.
                                       TL_TRANS   THE ARRAYED DATA STRUCTURE FOR TECNT, THE NUMBER OF
                                                  EXTERNAL NODE TARGETS IN TWO LEVEL STATE TRANSITIONS,
                                                  AND TSPRS, THE TRANSITION PAIRS.
                                       TRAN_MATRIX THE ARRAYED STATE TRANSITION MODEL COMPRISED OF THE
                                                  DIFFERENCE EQUATION WITH THE FIRST 224 POSITIONS USED
                                                  AS LIST HEADERS.  THE DATA STRUCTURE COMPRISES THE
                                                  DATA ELEMENTS LINK, THE LINK TO THE NEXT STATE, AND
                                                  PROB, THE PROBABLITY OF THIS TRANSITION OCCURRING.
                                    SCALARS:
                                       AVAIL      THE INDEX OF THE AVAILABLE SPACE LIST.
                                       CONV       THE SUM OF THE DIFFERENCES BETWEEN THE OLD PROBABLITIES
                                                  AND THE NEW PROBABILITIES.
                                       ECOUNT     THE EXTERNAL NODE COUNT FOR A GIVEN THIRD LEVEL STATE.
                                       I          THE SOURCE STATE OF THE TRANSITION.
```

```
 STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

                                   IT         THE INNER LOOP INDEX FOR PROCESSING A MAXIMUM OF
                                              THREE TRANSITIONS FROM TWO-LEVEL SUBTREES.
                                   ITRAN      THE TARGET STATE OF THE TRANSITIONS.
                                   J,K        INDICES.
                                   L          LEFT BRANCH.
                                   LNEWL,LNEWR THE NEW LEFT BRANCHES FROM A TRANSITION.
                                   LR         THE INDEX FOR A GIVEN PAIR OF LEFT AND RIGHT COMBINATIONS.
                                   M          THE MIDDLE BRANCH.
                                   PREV       THE PREVIOUS STATE(THE SOURCE STATE).
                                   PROB       THE INDICATED PROBABILITY.
                                   PROBX      THE PROBABILITY OF THE TRANSITION.
                                   PSUM       THE SUM OF THE PROBABILITIES ASSOCIATED WITH THE
                                              DIFFERENCE EQUATIONS OF EACH STATE.
                                   R          THE RIGHT BRANCH.
                                   RL         THE LEFT AND RIGHT COMBINATION.
                                   RNEWL,RNEWR THE NEW RIGHT BRANCH.
                                   SUM        THE SUM OF THE PROBABILITIES USED TO NORMALIZE THE
                                              PROBABILITY VECTOR.
                                   ZERO       THE VALUE ZERO.
                              */
                              /******* BEGIN GLOBAL DATA STRUCTURES *******/
    2    1          1         DCL LRCMB(28,2) FIXED BINARY STATIC INITIAL (
                                1,1, 2,1, 2,2, 3,1, 3,2, 3,3, 4,1, 4,2, 4,3, 4,4,
                                5,1, 5,2, 5,3, 5,4, 5,5, 6,1, 6,2, 6,3, 6,4, 6,5, 6,6,
                                7,1, 7,2, 7,3, 7,4, 7,5, 7,6, 7,7);
    3    1          1         DCL 1 TL_TRANS STATIC,
                                    2 TECNT(7,3) FLOAT(16) INITIAL(
                                      4.0E0,0.0E0,0.0E0,
                                      3.0E0,2.0E0,0.0E0,
                                      6.0E0,0.0E0,0.0E0,
                                      6.0E0,0.0E0,0.0E0,
                                      3.0E0,4.0E0,0.0E0,
                                      3.0E0,3.0E0,2.0E0,
                                      (3)3.0E0),
                                    2 TSPRS(7,3,2) FIXED BINARY INITIAL(
                                      2,0, 0,0, 0,0,
                                      4,0, 3,0, 0,0,
                                      5,0, 0,0, 0,0,
                                      5,0, 0,0, 0,0,
                                      1,1, 6,0, 0,0,
                                      1,2, 2,1, 7,0,
                                      2,2, 1,3, 3,1);
    4    1          1         DCL ETAB(0:7) FIXED BINARY STATIC INITIAL(0,4,5,6,6,7,8,9);
    5    1          1         DCL 1 TRAN_MATRIX(1500) STATIC,
                                    2 LINK FIXED BINARY,
                                    2 STATE FIXED BINARY,
                                    2 PROB FLOAT(16),
                                  AVAIL FIXED BINARY STATIC INITIAL(225);
    6    1          1         DCL P(224) FLOAT(16) STATIC;
                              /******* END GLOBAL DECLARATIONS *******/
                              /* INDEXING VARIABLES FOR MAIN PROGRAM */
    7    1          1         DCL (K,I,L,M,R,LR) FIXED BINARY STATIC;
                              /*
                              DCL PUNCH FILE STREAM OUTPUT;
                              */
```

```
 STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

                                    /*
                                      INITIALIZE THE FIRST 224 POSITION FOR THE EQUATION HEADERS.
                                    */
   8    1          1                  DO K=1 TO 224;
                                        /* USE CIRCULAR LISTS */
   9    1    1     1                    LINK(K) = K; STATE(K) = K; PROB(K) = 0.0E0;
  12    1    1     1                  END;
                                      /* GENERATE TRANSITIONS FROM STATES 1 TO 28 */
  13    1          1                  DO K = 1 TO 28;
  14    1    1     1                    CALL GENT1(K) ;
  15    1    1     1                  END;
                                      /* GENERATE TRANSITIONS FROM STATES 29 TO 224 */
  16    1          1                  DO K = 29 TO 224;
  17    1    1     1                    CALL GENT2(K);
  18    1    1     1                  END;
                                      /* ADJUST EQUATIONS FOR THE SAME STATE TRANSITION.  EXTERNAL NODE
                                         COUNT ON TRANSITION AND FIXED TOTAL EXTERNAL NODE COUNT */
  19    1          1                  CALL ADJUST;
  20    1          1                  CALL ECHECK(P);
                                      /* ITERATE EQUATIONS TO CONVERGENCE */
  21    1          1                  CALL SOLVE(P);
                                      /* OUTPUT SOLUTION VECTOR FOR FURTHER PROCESSING */
                                    /*PUT FILE (PUNCH) EDIT(P) ( COL(1),(3)E(25,15)); */
                                    /*
                                      PUT FILE (PUNCH) EDIT(P) (COL(1),(3)E(25,15));
                                    */
  22    1          1                   RETURN;
                                    /******* BEGIN SUPPORTING INTERNAL PROCEDURE *******/
                                    /*
                                    THIS ROUTINE DETERMINES THE STATE FROM THE LEFT, RIGHT, AND MIDDLE
                                    BRANCHES OF THE THIRD LEVEL NODE.
                                    */
  23    1          1                MAP: PROCEDURE(M,L,R) RETURNS(FIXED BINARY);
  24    2          2                 DCL (M,L,R) FIXED BINARY;
                                     /* MAP M,L,R TRIPLE INTO A 3RD LEVEL STATE IDENTIFIER   */
  25    2          2                  RETURN(28*M + (L*(L-1))/2 + R);
  26    2          2                END MAP;
                                    /*
                                    THIS ROUTINE GENERATES STATE TRANSITION FOR STATES  1 TO 28 FOR THIRD LEVEL
                                    NODES HAVING ONE KEY.  THE EXTERNAL NODE COUNT IS NOT INCLUDED IN THIS
                                    SEGMENT OF THE COMPUTATION NOR ARE THE SAME STATE TRANSITIONS.
                                    */
  27    1          1                GENT1: PROCEDURE(I);
  28    2          3                 DCL (ECOUNT, PROB) FLOAT(16);
  29    2          3                 DCL (I,LR,RL,IT,M,L,R,K) FIXED BINARY ;
  30    2          3                  IF I>28 THEN RETURN;
  32    2          3                  DO K=1 TO 2;
  33    2    1     3                    LR = LRCMB(I,K); RL = LRCMB(I,3-K);
                                    /*
                                        DETERMINE THE EXTERNAL NODE COUNT FROM THE EXTERNAL NODES
                                        DESCENDING FROM THE LEFT AND RIGHT BRANCHES.
                                    */
  35    2    1     3                    ECOUNT = ETAB(LR) + ETAB(RL);
                                    /*
                                        MAP THE TRANSITIONS.
```

```
 STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

                                       */
  36    2    1     3                     DO IT = 1 TO 3 WHILE(TSPRS(LR,IT,1)>0);
  37    2    2     3                       M = TSPRS(LR,IT,2);
  38    2    2     3                       L = MAX(TSPRS(LR,IT,1),RL);
  39    2    2     3                       R = MIN(TSPRS(LR,IT,1),RL);
  40    2    2     3                       PROB = TECNT(LR,IT)/ECOUNT;
                                       /*
                                           MAP THE TRANSITIONS ONTO THE TRANSITION MATRIX.
                                       */
  41    2    2     3                       CALL TMATX(MAP(M,L,R),I,PROB);
  42    2    2     3                     END;
  43    2    1     3                   END;
  44    2          3                   RETURN;
  45    2          3                  END GENT1;
                                       /*
                                       THIS ROUTINE GENERATES STATE TRANSITIONS UPON INSERTION FOR STATES 29 TO
                                       224 FOR THIRD LEVEL NODES HAVING TWO KEYS.  THE EXTERNAL NODE COUNT IS
                                       NOT INCLUDED NOR ARE THE SAME STATE TRANSTIONS.
                                       */
  46    1          1                  GENT2: PROCEDURE(I);
  47    2          4                  DCL ECOUNT FLOAT(16);
  48    2          4                  DCL (M,L,R,I,IT,RNEWL,LNEWR,LNEWL,RNEWR) FIXED BINARY,
                                            PROB FLOAT(16), ZERO FIXED BINARY STATIC INITIAL(0);
                                         /* COMPUTE M,L,R ACCORDING TO I */
  49    2          4                     M = (I-1)/28; IF M=0 THEN RETURN;
  52    2          4                     LR = MOD(I-1,28)+1; L = LRCMB(LR,1);  R = LRCMB(LR,2);
                                       /*
                                         DETERMINE THE EXTERNAL NODE COUNT.
                                       */
  55    2          4                     ECOUNT = ETAB(M)+ETAB(L) + ETAB(R);
                                         /* USE M   */
  56    2          4                     DO IT = 1 TO 3 WHILE(TSPRS(M,IT,1)>0);
  57    2    1     4                       PROB = TECNT(M,IT)/ECOUNT;
  58    2    1     4                       RNEWL = TSPRS(M,IT,1); LNEWR = TSPRS(M,IT,2);
  60    2    1     4                       IF LNEWR  =0 THEN CALL TMATX(MAP(RNEWL,L,R),I,PROB);
  62    2    1     4                       ELSE
                                             DO;  /* 3RD-LEVEL SPLIT */
  63    2    2     4                           CALL TMATX(MAP(ZERO,MAX(L,RNEWL),MIN(L,RNEWL)),I,PROB);
  64    2    2     4                           CALL TMATX(MAP(ZERO,MAX(R,LNEWR),MIN(R,LNEWR)),I,PROB);
  65    2    2     4                         END;
  66    2    1     4                     END;
                                         /* USE L*/
  67    2          4                     DO IT = 1 TO 3 WHILE(TSPRS(L,IT,1)>0);
  68    2    1     4                       PROB = TECNT(L,IT)/ECOUNT;
  69    2    1     4                       LNEWL = TSPRS(L,IT,1);  RNEWL = TSPRS(L,IT,2);
  71    2    1     4                       IF RNEWL=0 THEN CALL TMATX(MAP(M,MAX(LNEWL,R),MIN(LNEWL,R)),I,PROB)
                                             ;
  73    2    1     4                       ELSE
                                             DO;    /* 3RD-LEVEL SPLIT */
  74    2    2     4                           CALL TMATX(MAP(ZERO,MAX(M,R),MIN(M,R)),I,PROB);
  75    2    2     4                           CALL TMATX(MAP(ZERO,MAX(LNEWL,RNEWL),MIN(LNEWL,RNEWL)),I,PROB);
  76    2    2     4                         END;
  77    2    1     4                     END;
                                         /* USE R */
  78    2          4                     DO IT = 1 TO 3 WHILE(TSPRS(R,IT,1)>0);
```

```
 STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

  79    2    1    4                PROB = TECNT(R,IT)/ECOUNT;
  80    2    1    4                LNEWR  = TSPRS(R,IT,1);  RNEWR=TSPRS(R,IT,2);
  82    2    1    4                IF RNEWR=0 THEN CALL TMATX(MAP(M,MAX(L,LNEWR),MIN(L,LNEWR)),I,PROB)
                                   ;
  84    2    1    4                ELSE
                                   DO:  /* 3RD-LEVEL SPLIT */
  85    2    2    4                  CALL TMATX(MAP(ZERO,MAX(M,L),MIN(M,L)),I,PROB);
  86    2    2    4                  CALL TMATX(MAP(ZERO,MAX(LNEWR,RNEWR),MIN(LNEWR,RNEWR)),I,PROB);
  87    2    2    4                END;
  88    2    1    4              END;
  89    2         4              RETURN;
  90    2         4            END CENT2;
                                /*
                                THIS ROUTINE MAPS THE TRANSITIONS FROM THE SOURCE STATE TO THE TARGET
                                STATE.
                                 */
  91    1         1            TMATX: PROCEDURE(ITRAN,I,PROBX);
  92    2         5            DCL(ITRAN,I,K) FIXED BINARY, PROBX FLOAT(16);
                                /*
                                 ITRAN - TARGET STATE OF TRANSITION
                                 I -    SOURCE STATE OF TRANSITION
                                 PROBX - PROBABILITY OF TRANSITION
                                 ENTER (I,PROBX) INTO THE EQUATION LIST FOR STATE ITRAN
                                */
  93    2         5             DCL PREV FIXED BINARY;
  94    2         5              PREV = ITRAN;
  95    2         5              K = LINK(ITRAN);
  96    2         5              DO WHILE(K¬=ITRAN);
  97    2    1    5                IF I=STATE(K) THEN
  98    2    1    5                DO;  /* ALLOW FOR DUPLICATES DUE TO IDENICAL TWIN SPLIT */
  99    2    2    5                  PROB(K) = PROB(K) + PROBX; RETURN;
 101    2    2    5                END;
 102    2    1    5                IF I< STATE(K) THEN GO TO INSERT;
 104    2    1    5                PREV = K;
 105    2    1    5                K = LINK(K);
 106    2    1    5              END;
 107    2         5             INSERT:
                                 IF AVAIL > 1500 THEN
 108    2         5                DO; PUT SKIP LIST('***LIST MATRIX OVERFLOW***'); STOP; END;
 112    2         5              LINK(AVAIL) = K;
 113    2         5              LINK(PREV) = AVAIL;
 114    2         5              PROB(AVAIL) = PROBX;  STATE(AVAIL) = I;
 116    2         5              AVAIL = AVAIL + 1;
 117    2         5              RETURN;
 118    2         5            END TMATX;
                                /*
                                 SET UP SAME STATE TRANSITION PROBABILITIES AND ADJUST THE REST FOR
                                 EXTERNAL COUNTS FOR TARGET STATE.
                                 FIX TOTAL EXTERNAL NODE COUNT AT REASONABLY LOW VALUE (30) TO ASSURE
                                 REASONABLE CONVERGENCE RATE FOR DETERMINING LEFT EIGENVECTOR (
                                 ASYMPTOTIC STATE PROBABILITIES).
                                 */
 119    1         1            ADJUST: PROCEDURE;
 120    2         6              DCL (I,M,L,R,LR,K) FIXED BINARY, ECOUNT FLOAT(16);
 121    2         6               DO I=1 TO 224;
```

```
STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

 122    2     1     6          M = (I-1)/28; LR=MOD(I-1,28)+1; L = LRCMB(LR,1); R=LRCMB(LR,2);
 126    2     1     6          ECOUNT = ETAB(L) + ETAB(R) + ETAB(M);
 127    2     1     6          PROB(I) = (29.0E0-ECOUNT)/30.0E0;
 128    2     1     6          K = LINK(I);
 129    2     1     6          DO WHILE(K¬=I);
 130    2     2     6            PROB(K)=PROB(K)*ECOUNT/30.0E0;
 131    2     2     6            K = LINK(K);
 132    2     2     6          END;
 133    2     1     6        END;
 134    2           6      END ADJUST;
                            /*
                             CHECK TO SEE THAT THE PROBABLITIES OF TRANSITION FROM GIVEN STATE
                             (INCLUDING SAME STATE) ADD TO 1.0
                            */
 135    1           1      ECHECK: PROCEDURE(P);
 136    2           7       DCL P(*) FLOAT(16),(I,K) FIXED BINARY;
                             /* INITIALIZE TO SAME STATE */
 137    2           7        DO I = 1 TO 224;
 138    2     1     7          P(I) = PROB(I);
 139    2     1     7        END;
                             /* ADD IN THE REMAINING TRANSITION PROBABLITIES */
 140    2           7        DO I=1 TO 224;
 141    2     1     7          K = LINK(I);
 142    2     1     7          DO WHILE(K¬=I);
 143    2     2     7            P(STATE(K))=P(STATE(K))+PROB(K);
 144    2     2     7            K = LINK(K);
 145    2     2     7          END;
 146    2     1     7        END;
 147    2           7        PUT PAGE LIST('CONSISTENCY CHECK');
 148    2           7        PUT SKIP(3) LIST(P);
 149    2           7        RETURN;
 150    2           7      END ECHECK;
                           /*
                           THIS ROUTINE DETERMINES A SOLUTION.
                           */
 151    1           1      SOLVE: PROCEDURE(P);
 152    2           8       DCL (P(*),Q(224)) FLOAT(16), (P_INITIAL, CONV) FLOAT(16);
 153    2           8       DCL (I,J,K) FIXED BINARY; DCL SUM FLOAT(16);
                           /*GET LIST(P);*//* INITIALIZE TO INITIAL ESTIMATE */
                             /* NORMALIZE ESTIMATE */
 155    2           8        SUM = P(1);
 156    2           8        DO I=2 TO 224;
 157    2     1     8          SUM = SUM+P(I);
 158    2     1     8        END;
 159    2           8        DO I=1 TO 224;
 160    2     1     8          P(I) = P(I)/SUM;
 161    2     1     8        END;
 162    2           8        DO I=1 TO 10;
 163    2     1     8          DO J=1 TO 50;
 164    2     2     8          CALL ITER(P,Q,CONV);
 165    2     2     8          CALL ITER(Q,P,CONV);
 166    2     2     8          END;
 167    2     1     8          PUT PAGE LIST('CONVERGENCE VALUE',CONV);
 168    2     1     8          PUT SKIP(3) LIST(P);
 169    2     1     8        END;
```

```
 STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

  170    2            8          RETURN;
  171    2            8        END SOLVE;
                               /*
                               THIS ROUTINE COMPUTES NEW PROBABILITIES FROM THE RELATION THAT THE
                               PRODUCT OF THE PREVIOUS APPROXIMATION AND THE TRANSTION MATRIX
                               DETERMINE THE NEW APPROXIMATION FOR THE PROBABILITY VECTOR.
                               */
  172    1            1        ITER: PROCEDURE(POLD,PNEW,CONV);
  173    2            9         DCL (POLD(*),PNEW(*)) FLOAT(16), (CONV,PSUM) FLOAT(16);
  174    2            9         DCL (I,J,K) FIXED BINARY;
  175    2            9          CONV = 0.0E0;
  176    2            9          DO I=1 TO 224;
  177    2    1       9            PSUM = PROB(I)*POLD(I);
  178    2    1       9            K = LINK(I);
  179    2    1       9            DO WHILE (K¬=I);
  180    2    2       9              J = STATE(K);
  181    2    2       9              PSUM = PSUM+PROB(K)*POLD(J);
  182    2    2       9              K = LINK(K);
  183    2    2       9            END;
  184    2    1       9            PNEW(I) = PSUM;
  185    2    1       9            CONV = CONV+ABS(POLD(I)-PSUM);
  186    2    1       9          END;
  187    2            9          RETURN;
  188    2            9        END ITER;
                               /******* END SUPPORTING INTERNAL PROCEDURES *******/
  189    1            1        END S2_3L3;
    WARNING: FEATURES INCOMPATIBLE WITH PL/I-F HAVE BEEN USED (SY40)
```

CONSISTENCY CHECK

9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999998E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999998E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01	9.999999999999999E-01
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9.999999999999998E-01	9.999999999999998E-01	9.999999999999998E-01	9.999999999999998E-01	9.999999999999998E-01
9.999999999999998E-01	9.999999999999999E-01	9.999999999999998E-01	9.999999999999999E-01	9.999999999999999E-01
9.999999999999998E-01	9.999999999999999E-01	9.999999999999998E-01	9.999999999999998E-01	9.999999999999998E-01
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9.999999999999998E-01	9.999999999999999E-01	9.999999999999998E-01	9.999999999999998E-01	9.999999999999999E-01
9.999999999999999E-01	9.999999999999998E-01	9.999999999999998E-01	9.999999999999999E-01	

CONVERGENCE VALUE 1.628660983989456E-06

3.487107284109969E-02	7.224669560317693E-02	3.773980580565040E-02	2.503091820429628E-02	2.638488911872240E-02
4.917227440282499E-03	2.785250112942319E-02	3.527670049333817E-02	1.384857307361163E-02	9.711706553466795E-03
3.688878651870137E-02	4.860208161896617E-02	2.019796863725729E-02	2.890770021240002E-02	2.259877120044987E-02
1.585754024772076E-02	2.230879429057116E-02	9.721641954020332E-03	1.401588408402379E-02	2.293795185113956E-02
6.057345513070080E-03	3.172128607572798E-03	4.564572214554893E-03	2.053904872794786E-03	2.967684285278227E-03
5.023859355621993E-03	2.737623359143524E-03	3.175609609598293E-04	9.286672372118166E-03	2.075918033509746E-02
1.076661394365222E-02	8.014386784716780E-03	7.900656019946388E-03	1.378078329166841E-03	1.069751303655969E-02
1.000700986643003E-02	3.301601428760816E-03	1.883680813357013E-03	1.659310044765146E-02	1.522530317598382E-02
4.832546753445346E-03	5.199756477456641E-03	3.354460276276025E-03	8.483563382476778E-03	7.743263717917153E-03
2.397938859306361E-03	2.453622799226686E-03	2.952654556338835E-03	5.920093261110731E-04	1.891180317345285E-03
1.742277114157318E-03	5.375841769097354E-04	5.348541338135982E-04	6.185419524961588E-04	2.311041988950629E-04
2.105284487067351E-05	6.555665820607300E-03	1.521516211285434E-02	7.876430115450736E-03	6.137489784668870E-03
5.886773325163197E-03	1.020349633153421E-03	8.737750257481641E-03	8.228471328638186E-03	2.793420956888918E-03
1.901899724402218E-03	1.408367895173200E-02	1.280901229433951E-02	4.121223182746562E-03	5.506407137079711E-03
3.756910336126244E-03	7.473712592560023E-03	6.704975663721419E-03	2.084014348751789E-03	2.742070368079647E-03
3.549984601506920E-03	7.907402586909503E-04	1.712234216126331E-03	1.537594723516577E-03	4.708539020651678E-04
6.133439835656474E-04	7.674201430839530E-04	3.259923437805088E-04	3.206893221557238E-05	1.624558650106663E-03
3.825051380465587E-03	1.943437188352788E-03	1.636826087381602E-03	1.541492501195629E-03	2.855308615475845E-04
2.375145289048869E-03	2.222294546354755E-03	8.167391841639328E-04	5.835052719267998E-04	3.973219187606354E-03
3.571787035641413E-03	1.251509171552007E-03	1.774950455152881E-03	1.286841543384247E-03	2.176342728357469E-03
1.921240966679910E-03	6.524565645353748E-04	9.196796003074969E-04	1.280397443480610E-03	3.053138064320389E-04
5.110222974973313E-04	4.492101451632821E-04	1.501035613611989E-04	2.106522891912995E-04	2.851405301068613E-04
1.312821447615660E-04	1.363617928516512E-05	1.411994759547729E-03	3.813061698630793E-03	2.439637676975126E-03
1.633801667911621E-03	2.004240872966627E-03	3.939682415791924E-04	2.362516649718316E-03	2.878989976205946E-03
1.122965558182311E-03	7.992811949647332E-04	3.909361032030351E-03	4.645671365542235E-03	1.748020636134604E-03
2.469226746073577E-03	1.833287399817508E-03	2.120915638544627E-03	2.490121474592388E-03	9.134993221374888E-04
1.281874596753601E-03	1.836941001210783E-03	4.427292728443804E-04	4.949782326044088E-04	5.800958959704368E-04
2.099871913246220E-04	2.932389493115876E-04	4.097302726756027E-04	1.911345802516256E-04	1.997104693918368E-05
1.220050012230060E-03	3.437959693853213E-03	2.377664593496259E-03	1.570126121804694E-03	2.065576113175405E-03
4.332072093710193E-04	2.295589718267236E-03	3.012415660657751E-03	1.259264904518062E-03	9.148468809766661E-04
3.928229418473423E-03	5.039513392387992E-03	2.045623004438679E-03	2.963264611287192E-03	2.328127512145786E-03
2.190090674325568E-03	2.776261512780670E-03	1.104152521543787E-03	1.593660570808379E-03	2.437561789690403E-03
6.202335407967134E-04	5.221134486336622E-04	6.594357477771323E-04	2.590663984963109E-04	3.729944405433040E-04
5.591711461469233E-04	2.776519381286602E-04	3.032627495413544E-05	3.062102611472989E-04	9.681489690013105E-04
7.458169770366165E-04	4.579029721778847E-04	6.901837281675046E-04	1.560493001578906E-04	6.735690538294815E-04
1.014326165448845E-03	4.579561468556626E-04	3.359567783314560E-04	1.187591173002434E-03	1.761170031938701E-03
7.793415881164129E-04	1.141428321787609E-03	9.490963495394030E-04	6.772273898383503E-04	9.955203254454268E-04
4.338486894323232E-04	6.343478697034996E-04	1.034652631552164E-03	2.762923023410259E-04	1.643198929334818E-04
2.407715393089850E-04	1.039133681697429E-04	1.517200186016431E-04	2.437462393259390E-04	1.273344462025140E-04
1.452045289532988E-05	3.614996340936122E-05	1.240216997752410E-04	1.049073042754880E-04	6.142529331206566E-05
1.025408019579051E-04	2.468344255255552E-05	9.073558762446825E-05	1.514298391800340E-04	7.285017183977099E-05
5.375025361680965E-05	1.648188014747075E-04	2.723033698988800E-04	1.292020012165469E-04	1.904771722736925E-04
1.662376981544104E-04	9.610931868182591E-05	1.578109257298131E-04	7.405803138323282E-05	1.090765812332974E-04
1.877420251514311E-04	5.222544298601447E-05	2.372425145007693E-05	3.865848874814968E-05	1.810046815612729E-05
2.663643821568937E-05	4.532824442337779E-05	2.487843934676392E-05	2.922449638731729E-06	

CONVERGENCE VALUE 3.209921315655422E-11

3.487096805198799E-02	7.224666556684927E-02	3.773990184962035E-02	2.503097174499479E-02	2.638501132538209E-02
4.917254421149374E-03	2.785257547399146E-02	3.527686701430102E-02	1.384864817880787E-02	9.711758516762551E-03
3.688896745242874E-02	4.890236086159329E-02	2.019307097968857E-02	2.890784153111806E-02	2.259884610381113E-02
1.565764579179377E-02	2.230892396995535E-02	9.721679439790614E-03	1.401593424037027E-02	2.293797975751676E-02
6.057333003813229E-03	3.172152980564684E-03	4.564595521519060E-03	2.053909106188190E-03	2.967639332658435E-03
5.023853877356088E-03	2.737613690898907E-03	3.175592164943093E-04	9.286733064942450E-03	2.075928868093862E-02
1.076654088735690E-02	8.014407638524501E-03	7.900648593597448E-03	1.378072125960422E-03	1.069753573877903E-02
1.000699493392498E-02	3.301584542785770E-03	1.833669634760302E-03	1.659808506271585E-02	1.522523235304939E-02
4.932509670305643E-03	5.199712080486703E-03	3.354427086960192E-03	6.483531264324230E-03	7.743209733909183E-03
2.397917603727659E-03	2.453599366656969E-03	2.952627366109012E-03	5.920054868903502E-04	1.891168737428731E-03
1.742262670083965E-03	5.375794474511499E-04	5.348493003393192E-04	6.185377051666169E-04	2.311037002769619E-04
2.195292507444792E-03	6.555704198261898E-03	1.521520137692667E-02	7.876416671532133E-03	6.137482475581935E-03
5.886739971135616E-03	1.020340498602207E-03	8.737737711047600E-03	8.228423442752388E-03	2.793395432379727E-03
1.901871981625360E-03	1.408361415903730E-02	1.280690548364356E-02	4.121181263370235E-03	5.506350528406742E-03
3.756875230005186E-03	7.473660992463833E-03	6.704913394582152E-03	2.083995063510847E-03	2.742045154087342E-03
3.549962536529892E-03	7.907393498331013E-04	1.712220085898209E-03	1.537580969767487E-03	4.708506193336606E-04
6.133448831031965E-04	7.674186978140375E-04	3.259938557945306E-04	3.206929282091705E-05	1.624564834673935E-03
3.825047410796061E-03	1.948424971884997E-03	1.636818013705140E-03	1.541478062704346E-03	2.355277653456841E-04
2.375133217596739E-03	2.222736215626648E-03	8.167303446632865E-04	5.834989687567150E-04	3.973189079176796E-03
3.571750442763705E-03	1.251497207221002E-03	1.774933665137523E-03	1.286833869255780E-03	2.176323680342455E-03
1.921223173120085E-03	6.524523960401842E-04	9.196739070940107E-04	1.280396433972276E-03	3.053155643128607E-04
5.110178160977203E-04	4.492069949549024E-04	1.501032282434553E-04	2.106518362383987E-04	2.851419066106228E-04
1.312837212527239E-04	1.363642552149154E-05	1.411997827447918E-03	3.813052634547966E-03	2.439621067091115E-03
1.633791548101167E-03	2.004221515224506E-03	3.939539674045273E-04	2.362501721702891E-03	2.378962019206690E-03
1.122953398942507E-03	7.992725574379370E-04	3.909327207217583E-03	4.645623147823475E-03	1.748004121227247E-03
2.469203694633338E-03	1.833276821129882E-03	2.120895518627813E-03	2.490093561392221E-03	9.134936740139104E-04
1.281866963538034E-03	1.836939968145168E-03	4.427319135776830E-04	4.949738183310337E-04	5.300919952153069E-04
2.099567388963823E-04	2.932384904276859E-04	4.097323592138751E-04	1.911369116324461E-04	1.997141015735741E-05
1.220047498301157E-03	3.487936035332211E-03	2.377640091687463E-03	1.570111112935413E-03	2.065553068552319E-03
4.332029836910691E-04	2.295567714021795E-03	3.012382212084359E-03	1.259252765857531E-03	9.148381668148801E-04
3.928139078144022E-03	5.039465503206295E-03	2.046511097635823E-03	2.963247832657212E-03	2.328127677754419E-03
2.190070440528359E-03	2.776244319643607E-03	1.104151814287137E-03	1.593659918512777E-03	2.437577691382352E-03
6.202418551463590E-04	5.221098036674417E-04	6.594344502297689E-04	2.590676560782669E-04	3.729963557492691E-04
5.591781365248213E-04	2.776572005608190E-04	3.032697977432109E-05	3.062075996003929E-04	9.681386748605020E-04
7.458082699311355E-04	4.578978093650074E-04	6.901769311661858E-04	1.560484108648291E-04	6.735614979853189E-04
1.014316290863297E-03	4.579536058972312E-04	3.359549643665990E-04	1.187579808202880E-03	1.761159928528961E-03
7.793418248354362E-04	1.141428870310656E-03	9.491034520470687E-04	6.772231074549459E-04	9.955197677702110E-04
4.338515734918775E-04	6.343522112799154E-04	1.034667244351307E-03	2.762978670238913E-04	1.643195328413524E-04
2.407727127899037E-04	1.039145619729696E-04	1.517219386760320E-04	2.437509553338152E-04	1.278375036686124E-04
1.452082383341375E-05	3.614955550989906E-05	1.240202114171752E-04	1.049062508952856E-04	6.142467996730325E-05
1.025402298520109E-04	2.463346115588353E-05	9.073469158893440E-05	1.514290151937311E-04	7.285023764092813E-05
5.375031013019400E-05	1.648178368621323E-04	2.723035003504068E-04	1.292030215593599E-04	1.904787050659929E-04
1.662402217005633E-04	9.610924728256598E-05	1.578119908143380E-04	7.405909457738780E-05	1.090781627732245E-04
1.877459397305442E-04	5.222674250533480E-05	2.372436091929559E-05	3.885897266659636E-05	1.810081913310219E-05
2.663695808743747E-05	4.532934508879364E-05	2.487908637197287E-05	2.922520812214910E-06	

CONVERGENCE VALUE 9.917303150843315E-16

3.487096805104030E-02	7.224666556887036E-02	3.773990185257535E-02	2.503097174686196E-02	2.638501132815236E-02
4.917254421654838E-03	2.735257547628329E-02	3.527536701800395E-02	1.334864818019819E-02	9.711758517711257E-03
3.688896745667760E-02	4.880236086635568E-02	2.019807098124030E-02	2.890784153321610E-02	2.259884610948690E-02
1.585764579391437E-02	2.230892397201827E-02	9.721679440178020E-03	1.401593424085078E-02	2.293797975711761E-02
6.057338003422035E-03	3.172152961016063E-03	4.564595521807304E-03	2.053909106179225E-03	2.967689332626704E-03
5.023853377039172E-03	2.737513690615192E-03	3.175592164527911E-04	9.286733066015750E-03	2.075928868235603E-02
1.076664088736775E-02	8.014407638567901E-03	7.900648593060195E-03	1.378072125782470E-03	1.069753573873864E-02
1.000699493315048E-02	3.301584542326583E-03	1.833669634475389E-03	1.559808506165813E-02	1.522523235609411E-02
4.832509660490762E-03	5.199712079553115E-03	3.354427085367615E-03	8.483531263873611E-03	7.743209732697420E-03
2.397917603324034E-03	2.453599366233701E-03	2.952627355715164E-03	5.920054863601829E-04	1.891166787155214E-03
1.742262669600320E-03	5.375794473731770E-04	5.348493002652747E-04	6.185377051258661E-04	2.311037002882454E-04
2.105292507801819E-05	6.555704198783065E-03	1.521520137693893E-02	7.876416671159532E-03	6.137482475135075E-03
5.886739970256131E-03	1.020340498409433E-03	8.737737710373345E-03	8.228423441503052E-03	2.793395431848998E-03
1.901871981461746E-03	1.408361415724357E-02	1.280890543137292E-02	4.121181262628143E-03	5.506350527422256E-03
3.756875229526902E-03	7.473660991302152E-03	6.704913393436662E-03	2.083995063230440E-03	2.742045153732619E-03
3.549962536383663E-03	7.907393498920069E-04	1.712220085620043E-03	1.537580969550441E-03	4.708506193009757E-04
6.133443830665490E-04	7.674186976568057E-04	3.259938588556126E-04	3.206929233056030E-05	1.624564834710777E-03
3.825047410512467E-03	1.948424971568251E-03	1.635818013479706E-03	1.541478062402674E-03	2.855277652917521E-04
2.375133217265901E-03	2.222273621123985E-03	8.167303445106870E-04	5.834989536488796E-04	3.973189078511573E-03
3.571750442116652E-03	1.251497207058443E-03	1.774933664913494E-03	1.286333869217313E-03	2.176323679978117E-03
1.921223172860543E-03	6.524523960108135E-04	9.196739090576856E-04	1.280396434060120E-03	3.053155643774300E-04
5.110178160231878E-04	4.492069949330688E-04	1.501032262506453E-04	2.106518862500024E-04	2.851419066652382E-04
1.312837212940977E-04	1.363642552692797E-05	1.411997327430203E-03	3.813052684169104E-03	2.439621066676398E-03
1.633791547842729E-03	2.004221514827653E-03	3.939539673309751E-04	2.362501721326678E-03	2.878962018638415E-03
1.122953398735363E-03	7.992725572922874E-04	3.909327206509386E-03	4.645623146988030E-03	1.748004121007346E-03
2.469203694332950E-03	1.833276321083832E-03	2.120895518263135E-03	2.490098561069394E-03	9.134936739779186E-04
1.281856953495630E-03	1.836939968306911E-03	4.427319136729945E-04	4.949738182620599E-04	5.800919951798663E-04
2.099867889104250E-04	2.932384904462283E-04	4.097323592944505E-04	1.911369116932616E-04	1.997141016535113E-05
1.220047498131643E-03	3.487936034732094E-03	2.377640091190061E-03	1.570111112624237E-03	2.065553068159459E-03
4.332029836362507E-04	2.295567713563683E-03	3.012352211513116E-03	1.259252765702297E-03	9.148381667050706E-04
3.928189077433746E-03	5.039465502572630E-03	2.046611096310212E-03	2.963247882600056E-03	2.328127678001447E-03
2.190070440234631E-03	2.776244319539278E-03	1.104151814334743E-03	1.593659918661581E-03	2.437577691937309E-03
6.202418553583159E-04	5.221098088311155E-04	6.594344502652111E-04	2.590676561280735E-04	3.729963558230140E-04
5.591781367060554E-04	2.776572006754095E-04	3.032697978784033E-05	3.062079995436300E-04	9.681366746517934E-04
7.458082697851813E-04	4.578575092763101E-04	6.901769310800833E-04	1.560484108624143E-04	6.735614978580251E-04
1.014316290744714E-03	4.579536058916719E-04	3.359549643636298E-04	1.187579808050932E-03	1.761159928493383E-03
7.793413249237692E-04	1.141428870444255E-03	9.491034522830353E-04	6.772231074256762E-04	9.955197673602267E-04
4.338515735921300E-04	6.343522114237541E-04	1.034667244718167E-03	2.762978671432043E-04	1.643195328493000E-04
2.407727128362874E-04	1.039146620966097E-04	1.517219387256510E-04	2.437509554360000E-04	1.278375037267011E-04
1.452082383959092E-05	3.614955550167860E-05	1.240202113922795E-04	1.049062508823614E-04	6.142467905951497E-05
1.025402298510110E-04	2.468346115900873E-05	9.073469157768017E-05	1.514290151926974E-04	7.285023765035715E-05
5.375031013730080E-05	1.648178368583975E-04	2.723035003825784E-04	1.292030215927632E-04	1.904787051157181E-04
1.662402217628598E-04	9.610924729087212E-05	1.578119908511667E-04	7.405909460402621E-05	1.090781628126879E-04
1.877459398138226E-04	5.222674252971558E-05	2.372436092374235E-05	3.885897267918024E-05	1.810081914071508E-05
2.663695809868043E-05	4.532934510961618E-05	2.487908638263555E-05	2.922520813187140E-06	

CONVERGENCE VALUE 8.839172087007256E-16

3.487096805103724E-02	7.224666556386408E-02	3.773990185257211E-02	2.503097174685931E-02	2.638501132815010E-02
4.917254421654414E-03	2.785257547623090E-02	3.527686701800094E-02	1.384864818019700E-02	9.711758517710422E-03
3.638896745667445E-02	4.880235086685169E-02	2.019807098123854E-02	2.890784153321359E-02	2.259884610948491E-02
1.585764579391351E-02	2.230892397201633E-02	9.721679440177164E-03	1.401593424084954E-02	2.293797975711557E-02
6.057333008421488E-03	3.172152981015790E-03	4.564595521306904E-03	2.053909106179042E-03	2.967689332626439E-03
5.023853877033722E-03	2.737613690514944E-03	3.175592164527622E-04	9.286733066015962E-03	2.075928368235421E-02
1.076664088736679E-02	3.014407638567190E-03	7.900648593059483E-03	1.378072125782344E-03	1.069753573873769E-02
1.000699493314958E-02	3.301584542328282E-03	1.883669634475217E-03	1.659808506165663E-02	1.522523235609273E-02
4.832509669490321E-03	5.199712079552640E-03	3.354427036367310E-03	8.483531263872843E-03	7.743209732696714E-03
2.397917603323816E-03	2.453599366233473E-03	2.952627365714900E-03	5.920054868501309E-04	1.891168787155042E-03
1.742262669800162E-03	5.375794473731285E-04	5.343493002652267E-04	6.185377051258117E-04	2.311037002832256E-04
2.105292507801643E-05	6.555704198782492E-03	1.521520137693758E-02	7.876416671158820E-03	6.137482475134522E-03
5.886739970255594E-03	1.020340498409340E-03	8.737737710378056E-03	8.223423441502302E-03	2.793395431848742E-03
1.901871981461572E-03	1.408361415724229E-02	1.280390548137175E-02	4.121181262627769E-03	5.506350527421766E-03
3.756875229526567E-03	7.473660991301469E-03	6.704913393436051E-03	2.083995063230253E-03	2.742045153732374E-03
3.549962536333357E-03	7.907393498919395E-04	1.712220085619387E-03	1.537560969550303E-03	4.708506193009341E-04
6.133448830664950E-04	7.674186978567399E-04	3.259933538555855E-04	3.206929283055768E-05	1.624564834710633E-03
3.825047410512122E-03	1.948424971568073E-03	1.636818013479557E-03	1.541478062402533E-03	2.855277652917261E-04
2.375133217265684E-03	2.222273621128781E-03	8.167303445106127E-04	5.834989686488266E-04	3.973189078511210E-03
3.571750442116327E-03	1.251497207058332E-03	1.774933665491335E-03	1.286833869217201E-03	2.176323679977919E-03
1.921223172860370E-03	6.524523960107564E-04	9.196739090576050E-04	1.280396434080011E-03	3.053155643774046E-04
5.110178160231437E-04	4.492069949330291E-04	1.501032282506324E-04	2.106518862499843E-04	2.851419066652144E-04
1.312837212940870E-04	1.363642552692687E-05	1.411997827430077E-03	3.813052684168758E-03	2.439621066676175E-03
1.633791547842579E-03	2.004221514327470E-03	3.939639673309394E-04	2.362501721326462E-03	2.878962018638152E-03
1.122953398735261E-03	7.992725572922148E-04	3.909327206509029E-03	4.645623146987657E-03	1.748004121007190E-03
2.469203674332730E-03	1.833276821083672E-03	2.120895518262941E-03	2.490098561069170E-03	9.134936739778386E-04
1.281866963495518E-03	1.830939968308755E-03	4.427319136729579E-04	4.949733182620154E-04	5.800919951798158E-04
2.099267889104100E-04	2.932384904462033E-04	4.097323592944165E-04	1.911369116932461E-04	1.997141016534952E-05
1.220047498181537E-03	3.487936034731774E-03	2.377640091189843E-03	1.570111112624093E-03	2.065553068159271E-03
4.332029336302120E-04	2.295567713568473E-03	3.012382211517842E-03	1.259252765702184E-03	9.148381667049892E-04
3.923189077433368E-03	5.033465502572179E-03	2.046511096310034E-03	2.963247802599798E-03	2.323127678001250E-03
2.190070440234434E-03	2.776244319539036E-03	1.104151814384649E-03	1.593659918561446E-03	2.437577691937107E-03
6.202418553582656E-04	5.2210998088310694E-04	6.594344502651546E-04	2.590676501280519E-04	3.729963558229830E-04
5.591781367060209E-04	2.776572006753871E-04	3.032697973783787E-05	3.062079995436019E-04	9.681386746517044E-04
7.458082697851136E-04	4.573978092767685E-04	6.901769310800218E-04	1.560484108524007E-04	6.735614978579639E-04
1.014316290744623E-03	4.579535058916322E-04	3.359549643636008E-04	1.187579808050826E-03	1.761159928493229E-03
7.793413249237034E-04	1.141428870444159E-03	9.491034522829572E-04	6.772231074256169E-04	9.955197678601422E-04
4.338515735920941E-04	6.343522114287018E-04	1.034667244718084E-03	2.762978671431820E-04	1.643195323492859E-04
2.407727129362673E-04	1.039146620066012E-04	1.517219387256387E-04	2.437509554359803E-04	1.278375037266907E-04
1.452082383953972E-05	3.614955550167527E-05	1.240202113922682E-04	1.049062508823520E-04	6.142467995950950E-05
1.025402298510021E-04	2.468346115900665E-05	9.073469157767209E-05	1.514290151926843E-04	7.285023765035103E-05
5.375031013729628E-05	1.648178368583832E-04	2.723035003825554E-04	1.292030215927526E-04	1.904787051157025E-04
1.662402217623464E-04	9.610924729086396E-05	1.578119908511537E-04	7.405909460402022E-05	1.090781628126791E-04
1.877459398133075E-04	5.222674252971135E-05	2.372436092374038E-05	3.885897267917708E-05	1.810081914071363E-05
2.663695809867829E-05	4.532934510961251E-05	2.487908638263349E-05	2.922520813186889E-06	

CONVERGENCE VALUE 8.562416776986122E-16

3.487096805103424E-02	7.224666556835794E-02	3.773990185256888E-02	2.503097174685767E-02	2.638501132814783E-02
4.917254421653993E-03	2.785257547527852E-02	3.527686701799791E-02	1.384864818019581E-02	9.711758517709589E-03
3.688896745667129E-02	4.880236086684751E-02	2.019807098123681E-02	2.890784153321111E-02	2.259884610948297E-02
1.585764579391216E-02	2.230892397201442E-02	9.721679440176333E-03	1.401593424084834E-02	2.293797975711360E-02
6.057338008420970E-03	3.172152981015518E-03	4.564595521806515E-03	2.053909106178867E-03	2.967689332626135E-03
5.023853877038269E-03	2.737613690614709E-03	3.175392164527349E-04	9.286733066015166E-03	2.075928863235244E-02
1.076664088736567E-02	8.014407638566503E-03	7.900648593058807E-03	1.378072125782226E-03	1.069753573373678E-02
1.000699493314872E-02	3.301584542327999E-03	1.883659634475056E-03	1.659808506165521E-02	1.522523235609142E-02
4.332509669489906E-03	5.199712079552195E-03	3.354427086367022E-03	8.483531263872115E-03	7.743209732696049E-03
2.397917603323610E-03	2.453599366233268E-03	2.952627365714646E-03	5.920054868600800E-04	1.891168787154879E-03
1.742262669300012E-03	5.375794473730823E-04	5.348493002651807E-04	6.185377051257565E-04	2.311037002882058E-04
2.105292507801452E-05	6.555704198781930E-03	1.521520137693628E-02	7.876416671158146E-03	6.137482475133995E-03
5.886739970255089E-03	1.020340498409252E-03	8.737737710377308E-03	8.228423441501596E-03	2.793395431848502E-03
1.901871981461409E-03	1.409361415724108E-02	1.280890548137065E-02	4.121181262627416E-03	5.506350527421294E-03
3.756875229526245E-03	7.473660991300828E-03	6.704913393435476E-03	2.083995063230074E-03	2.742045153732138E-03
3.549962536383053E-03	7.907393498918716E-04	1.712220085619740E-03	1.537580969550171E-03	4.708506193006937E-04
6.133443830664424E-04	7.674186978566740E-04	3.259933586555574E-04	3.206929283055493E-05	1.624564834710494E-03
3.825047410511794E-03	1.948424971567906E-03	1.636818013479417E-03	1.541478062402401E-03	2.855277652917016E-04
2.375133217265481E-03	2.222273621128591E-03	8.167303445105428E-04	5.834989685487766E-04	3.973189078510869E-03
3.571750442116021E-03	1.251497207053224E-03	1.774933664913183E-03	1.286833869217091E-03	2.176323679977732E-03
1.921223172860205E-03	6.524523960107003E-04	9.196739090575262E-04	1.280396434079901E-03	3.053155643773784E-04
5.110178160230997E-04	4.492069949329906E-04	1.501032282506195E-04	2.106518862499663E-04	2.851419066651900E-04
1.312837212940757E-04	1.363642552692570E-05	1.411997627429956E-03	3.813052634168431E-03	2.439621066675966E-03
1.633791547842439E-03	2.004221514827298E-03	3.939639673309055E-04	2.362501721326260E-03	2.878962018637905E-03
1.122953398735164E-03	7.992725572921464E-04	3.909327206508693E-03	4.645623146987258E-03	1.748004121007040E-03
2.469203694332518E-03	1.833276321083515E-03	2.120895518262759E-03	2.490098561068957E-03	9.134936739777603E-04
1.281866963495403E-03	1.836939568308597E-03	4.427319136729198E-04	4.949738132619728E-04	5.800919951797660E-04
2.099867889103920E-04	2.932384904461781E-04	4.097323592943813E-04	1.911369115932296E-04	1.997141016534780E-05
1.220047498181432E-03	3.437936034731475E-03	2.377640091169639E-03	1.570111112623958E-03	2.065553063159094E-03
4.332029836361743E-04	2.295567713563276E-03	3.012382211517584E-03	1.259252765702076E-03	9.148381667049106E-04
3.928189077433051E-03	5.039465502571746E-03	2.046611096309859E-03	2.963247882599544E-03	2.328127673001050E-03
2.190070440234247E-03	2.776244319538797E-03	1.104151814334554E-03	1.593659918661309E-03	2.437577691936898E-03
6.202418553562123E-04	5.221098088310246E-04	6.594344502650980E-04	2.590676561280297E-04	3.729963558229509E-04
5.591781367059728E-04	2.776572006753632E-04	3.032697978783526E-05	3.062079995435756E-04	9.681386746516214E-04
7.453082697850476E-04	4.578978092767292E-04	6.901769310799625E-04	1.560484103623873E-04	6.735614978579061E-04
1.014316290744536E-03	4.579536058915923E-04	3.359547643635719E-04	1.187579808050724E-03	1.761159928493078E-03
7.793418249236364E-04	1.141428870444061E-03	9.491034522328757E-04	6.772231074255588E-04	9.955197678600567E-04
4.338515735920569E-04	6.343522114286473E-04	1.034667244717995E-03	2.762978671431583E-04	1.643195328492718E-04
2.407727128362466E-04	1.039146620065923E-04	1.517219387256256E-04	2.437509554359594E-04	1.278375037266797E-04
1.452082383958847E-05	3.614955550167217E-05	1.240202113922575E-04	1.049062508823430E-04	6.142467995950422E-05
1.025402298509933E-04	2.463346115900453E-05	9.073469157766430E-05	1.514290151926713E-04	7.285023765034476E-05
5.375031013729166E-05	1.648178368583691E-04	2.723035003825321E-04	1.292030215927415E-04	1.904787051156861E-04
1.662402217623321E-04	9.610924729085571E-05	1.578119908511401E-04	7.405909460401386E-05	1.090781628126697E-04
1.877459398137914E-04	5.222674252970686E-05	2.372436092373834E-05	3.885897267917373E-05	1.810081914071207E-05
2.663695809867599E-05	4.532934510960861E-05	2.487908638263136E-05	2.922520813186638E-06	

CONVERGENCE VALUE 8.941516960441625E-16

3.487096805103118E-02	7.224666556885143E-02	3.773990135256551E-02	2.503097174685544E-02	2.638501132814549E-02
4.917254421653560E-03	2.785257547627603E-02	3.527686701799478E-02	1.384864813019459E-02	9.711758517708727E-03
3.688896745666802E-02	4.880236086684318E-02	2.019807098123502E-02	2.890734153320855E-02	2.259884610948097E-02
1.585764579391075E-02	2.230392397201245E-02	9.721679440175469E-03	1.401593424084710E-02	2.293797975711158E-02
6.057338098420436E-03	3.172152981015238E-03	4.564595521806112E-03	2.053909106173685E-03	2.967689332625924E-03
5.023853877037847E-03	2.737613690614468E-03	3.175592164527070E-04	9.286733066014344E-03	2.075928868235060E-02
1.076664083736492E-02	8.014407638565795E-03	7.900548593058107E-03	1.378072125782105E-03	1.069753573873583E-02
1.000699493314783E-02	3.301584542327707E-03	1.883669634474839E-03	1.659803506165375E-02	1.522523235609008E-02
4.832509669489481E-03	5.199712079551735E-03	3.354427086366727E-03	8.483531263371366E-03	7.743209732695367E-03
2.397917603323399E-03	2.453599366233051E-03	2.952627365714386E-03	5.920054868500231E-04	1.891168787154713E-03
1.742262669799359E-03	5.375794473730351E-04	5.348493002651337E-04	6.185377051257043E-04	2.311037002881855E-04
2.105292507801277E-05	6.555704198781351E-03	1.521520137593493E-02	7.876416671157449E-03	6.137482475133453E-03
5.885739970254569E-03	1.020340496409162E-03	3.737737710376536E-03	8.228423441500870E-03	2.793395431848256E-03
1.901871981461241E-03	1.498361415723933E-02	1.280890548136952E-02	4.121181262627052E-03	5.506350527420809E-03
3.756875229525914E-03	7.473660991300170E-03	6.704913393434886E-03	2.083995063229891E-03	2.742045153731897E-03
3.549962536382741E-03	7.907393498918022E-04	1.712220085619589E-03	1.537580969550036E-03	4.708506193008524E-04
6.133448830663336E-04	7.674186978566067E-04	3.259938588555289E-04	3.206929283055211E-05	1.624564834710350E-03
3.825047410511456E-03	1.948424971567734E-03	1.636318013479272E-03	1.541478062402265E-03	2.855277652916765E-04
2.375133217265271E-03	2.222273621128395E-03	8.167303445104708E-04	5.834989696487252E-04	3.973189078510520E-03
3.571750442115706E-03	1.251497207055114E-03	1.774933654913027E-03	1.286833869216977E-03	2.176323679977540E-03
1.921223172860036E-03	6.524523960106430E-04	9.196739090574453E-04	1.280396434079788E-03	3.053155643773516E-04
5.110178160230549E-04	4.492069949329511E-04	1.501032282500064E-04	2.106518362499478E-04	2.851419066651640E-04
1.312837212940642E-04	1.363642552692450E-05	1.411997827429331E-03	3.813052684168094E-03	2.439621066675750E-03
1.633791547842295E-03	2.004221514327121E-03	3.939639673308709E-04	2.362501721325051E-03	2.878962018637651E-03
1.122953398735065E-03	7.992725572920759E-04	3.909327206508349E-03	4.645623146986850E-03	1.748004121006886E-03
2.469203694332301E-03	1.833276321083354E-03	2.120895513262573E-03	2.490098561068738E-03	9.134936739776800E-04
1.281866963495295E-03	1.336939968308436E-03	4.427319136728809E-04	4.949738132619293E-04	5.800919951797150E-04
2.099367889103735E-04	2.932384904461523E-04	4.097323592943454E-04	1.911369116932129E-04	1.997141016534605E-05
1.220047498181325E-03	3.467936034731167E-03	2.377640091189430E-03	1.570111126238820E-03	2.065553068158912E-03
4.332029836361363E-04	2.295567713568074E-03	3.012382211517319E-03	1.259252765701966E-03	9.148381667048302E-04
3.928189077432706E-03	5.039465502571303E-03	2.046511095309670E-03	2.963247882599283E-03	2.328127673000846E-03
2.190070440234054E-03	2.776244319538553E-03	1.104151814384457E-03	1.593659918661159E-03	2.437577691936684E-03
6.202418553581580E-04	5.221098088309787E-04	6.594344502650401E-04	2.590676561280070E-04	3.729963558229182E-04
5.591781367059239E-04	2.776572006753389E-04	3.032697973783261E-05	3.062079995435486E-04	9.681386746515361E-04
7.458082697847340E-04	4.573978092766389E-04	6.901769310799020E-04	1.560484108523736E-04	6.735614978578468E-04
1.014316290744447E-03	4.579536058915526E-04	3.359549643635424E-04	1.187579808050620E-03	1.761159928492923E-03
7.793418249235680E-04	1.141428870443961E-03	9.491034522827925E-04	6.772231074254993E-04	9.955197678599693E-04
4.338515735920188E-04	6.343522114285917E-04	1.034667244717904E-03	2.762978671431341E-04	1.643195328492574E-04
2.407727128362255E-04	1.039146620065832E-04	1.517219387256123E-04	2.437509554359381E-04	1.278375037266685E-04
1.452082383958720E-05	3.614955501668968E-05	1.240202113922466E-04	1.049062508823338E-04	6.142467995949883E-05
1.025402298509843E-04	2.468346115900237E-05	9.073469157765634E-05	1.514290151926580E-04	7.285023765033838E-05
5.375031013728695E-05	1.648178368583546E-04	2.723035003825082E-04	1.292030215927302E-04	1.904737051156694E-04
1.662402217628176E-04	9.610924729084728E-05	1.578119908511263E-04	7.405909460400738E-05	1.090781628126601E-04
1.877459398137750E-04	5.222674252970230E-05	2.372436092373626E-05	3.885397267917033E-05	1.810081914071049E-05
2.663695809867366E-05	4.532934510960464E-05	2.487908638262918E-05	2.922520813186382E-06	

:ONVERGENCE VALUE 8.558202788073532E-16

3.487096805102802E-02	7.224666556884508E-02	3.773990185256214E-02	2.503097174585320E-02	2.638501132814311E-02
4.917254421653112E-03	2.735257547527355E-02	3.527685701799160E-02	1.384364818019334E-02	9.711753517707850E-03
3.688896745666469E-02	4.830236086683876E-02	2.019807098123320E-02	2.890784153320594E-02	2.259884610947893E-02
1.585764579390932E-02	2.230892397201043E-02	9.721679440174593E-03	1.401593424034533E-02	2.293797975710950E-02
6.057333008419887E-03	3.172152961014951E-03	4.564595521805696E-03	2.053909106178499E-03	2.967689332625654E-03
5.023853377037392E-03	2.737613490614220E-03	3.175592164526782E-04	9.286733066013504E-03	2.075928868234872E-02
1.076664088736394E-02	8.014407638565070E-03	7.900648593057393E-03	1.378072125781930E-03	1.069753573873486E-02
1.000699493314693E-02	3.301584542327409E-03	1.883659634474719E-03	1.659808506165224E-02	1.522523235608870E-02
4.832509669489043E-03	5.159712079551265E-03	3.354427086366423E-03	8.483531263870599E-03	7.743209732694667E-03
2.397917603323182E-03	2.453599366232830E-03	2.952627365714119E-03	5.920054863599745E-04	1.891168787154542E-03
1.742262669799701E-03	5.375794473729364E-04	5.348493002650853E-04	6.185377051256482E-04	2.311037002881645E-04
2.105292507801056E-05	6.555704198780759E-03	1.521520137693355E-02	7.876416671156737E-03	6.137482475132898E-03
5.386739970254037E-03	1.020340498409070E-03	8.737737710375745E-03	8.228423441500125E-03	2.793395431846003E-03
1.901871981461069E-03	1.408361415723856E-02	1.230890548136336E-02	4.121181262626679E-03	5.506350527420310E-03
3.756875229525574E-03	7.473660991299493E-03	6.704913393434278E-03	2.083995063229702E-03	2.742045153731649E-03
3.549962536332419E-03	7.907393498917305E-04	1.712220085619434E-03	1.537580969549896E-03	4.708506193008097E-04
6.133448830663329E-04	7.674186978565371E-04	3.259938588554993E-04	3.206929283054920E-05	1.624564834710203E-03
3.825047410511110E-03	1.948424971567558E-03	1.636818013479124E-03	1.541478062402126E-03	2.855277652916506E-04
2.375133217265056E-03	2.222736211281194E-03	8.167303445103969E-04	5.834989636486724E-04	3.973189078510160E-03
3.571750442115383E-03	1.251497207058001E-03	1.774933664912866E-03	1.286833869216861E-03	2.176323679977343E-03
1.921223172859862E-03	6.524523960105839E-04	9.196739090573619E-04	1.280396434079672E-03	3.053155643773239E-04
5.110178160230085E-04	4.492069949329104E-04	1.501032282505927E-04	2.106513862499286E-04	2.851419066651391E-04
1.312837212940523E-04	1.353642552692326E-05	1.411997827429704E-03	3.813052684167749E-03	2.439621066675530E-03
1.633791547842148E-03	2.004221514826940E-03	3.939639673308352E-04	2.362501721325838E-03	2.878962018637391E-03
1.122953398734964E-03	7.992725572920036E-04	3.909327206507996E-03	4.645623146986429E-03	1.748004121006728E-03
2.469203694332077E-03	1.833276821083187E-03	2.120395518262381E-03	2.490098561068512E-03	9.134936739775972E-04
1.281866963495179E-03	1.836939968308269E-03	4.427319136728408E-04	4.949738182518845E-04	5.800919951796624E-04
2.099367889103545E-04	2.932384904461257E-04	4.097323592943081E-04	1.911369116931955E-04	1.997141016534424E-05
1.220047498181214E-03	3.487936034730852E-03	2.377640091189215E-03	1.570111112623678E-03	2.065553068158725E-03
4.332029336360975E-04	2.295567713567866E-03	3.012382211517046E-03	1.259252765701852E-03	9.148381667047475E-04
3.928139077432349E-03	5.039465502570847E-03	2.046611096309493E-03	2.963247382599015E-03	2.328127678000634E-03
2.190070440233856E-03	2.776244319533302E-03	1.104151814384357E-03	1.593659918661025E-03	2.437577691936463E-03
6.202413553581016E-04	5.221098083309313E-04	6.594344502649803E-04	2.590676561279835E-04	3.729963558228843E-04
5.591781367058730E-04	2.776572006753137E-04	3.032697973782985E-05	3.062079995435209E-04	9.681386746514435E-04
7.458082697849165E-04	4.573978092766475E-04	6.901769310798395E-04	1.560484108623595E-04	6.735614976577859E-04
1.014316290744355E-03	4.579536058915111E-04	3.359549643635120E-04	1.187579808050512E-03	1.761159928492764E-03
7.793418249234974E-04	1.141428870443857E-03	9.491034522827062E-04	6.772231074254379E-04	9.955197673598791E-04
4.338515735919795E-04	6.343522114285341E-04	1.034557244717810E-03	2.762978671431090E-04	1.643195328492424E-04
2.407727128362037E-04	1.039146620065737E-04	1.517219387255985E-04	2.437509554359159E-04	1.278375037266569E-04
1.452082383958538E-05	3.614955550166571E-05	1.240202113922354E-04	1.049062503823243E-04	6.142467995949327E-05
1.025402298509750E-04	2.468346115900013E-05	9.073469157764812E-05	1.514290151926443E-04	7.285023765033177E-05
5.375031013728207E-05	1.648178368583396E-04	2.723035003824835E-04	1.292030215927185E-04	1.904787051156521E-04
1.662402217628024E-04	9.610924729083855E-05	1.573119908511119E-04	7.405909460400064E-05	1.090781628126502E-04
1.877459398137579E-04	5.222674252969754E-05	2.372436092373410E-05	3.885897267916680E-05	1.810081914070884E-05
2.663695309867124E-05	4.532934510960051E-05	2.487908638262691E-05	2.922520813186116E-06	

CONVERGENCE VALUE 8.742591272778953E-16

3.487096805102497E-02	7.224666556883869E-02	3.773990135255884E-02	2.503097174685101E-02	2.638501132814082E-02
4.917254421652688E-03	2.785257547627110E-02	3.527686701798854E-02	1.384864818019214E-02	9.711758517707009E-03
3.688896745666148E-02	4.880236086683454E-02	2.019807098123145E-02	2.890784153320343E-02	2.259884610947697E-02
1.585764579390794E-02	2.230892397200849E-02	9.721579440173749E-03	1.401593424084462E-02	2.293797975710751E-02
6.057338008419360E-03	3.172152981014674E-03	4.564595521805297E-03	2.053909106178320E-03	2.967689332625396E-03
5.023853877036956E-03	2.737613690613982E-03	3.175592164526505E-04	9.286733066012698E-03	2.075928868234692E-02
1.076664088736301E-02	8.014407638564374E-03	7.900648593056706E-03	1.378072125781860E-03	1.069753573873393E-02
1.000679493314606E-02	3.301564542327122E-03	1.883669634474555E-03	1.659808506165080E-02	1.522523235606738E-02
4.832509669438622E-03	5.199712079550313E-03	3.354427066366131E-03	8.483531263869863E-03	7.743209732693992E-03
2.397917603322973E-03	2.453599366232616E-03	2.952627365713862E-03	5.920054858599228E-04	1.891168787154377E-03
1.742262669799549E-03	5.375794473729395E-04	5.348493002650386E-04	6.185377051255942E-04	2.311037002831443E-04
2.105292507800902E-05	6.555704198780138E-03	1.521520137693223E-02	7.876416671156053E-03	6.137482475132366E-03
5.886739970253526E-03	1.020340498408981E-03	8.737737710374988E-03	8.228423441499411E-03	2.793395431847761E-03
1.901871981460904E-03	1.408361415723734E-02	1.280390543136725E-02	4.121181262626320E-03	5.506350527419331E-03
3.756875229525246E-03	7.473660991298643E-03	6.704913393433695E-03	2.063995063229521E-03	2.742045153731410E-03
3.549962536382109E-03	7.907393498916614E-04	1.712220085619285E-03	1.537580969549762E-03	4.708506193007686E-04
6.133448830662794E-04	7.674186978534701E-04	3.259933586554708E-04	3.206929233054640E-05	1.624564834710062E-03
3.825047410510777E-03	1.943424971567338E-03	1.636818013478982E-03	1.541478062401991E-03	2.855277652916258E-04
2.375133217264350E-03	2.222736211280000E-03	8.167303445103257E-04	5.834989636486216E-04	3.973189073509814E-03
3.571750442115072E-03	1.251497207057892E-03	1.774933664912711E-03	1.286833869216749E-03	2.176323679977154E-03
1.921223172859695E-03	6.524523960105269E-04	9.196739090572817E-04	1.280396434079561E-03	3.053155643772972E-04
5.110178160229639E-04	4.492069949328712E-04	1.501032282505796E-04	2.106518862499103E-04	2.851419066651142E-04
1.312837212940408E-04	1.363642552692207E-05	1.411997827429581E-03	3.813052684167413E-03	2.439621066675318E-03
1.633791547842006E-03	2.004221514826766E-03	3.939639673308009E-04	2.362501721325632E-03	2.878962018637140E-03
1.122953398734856E-03	7.992725572919340E-04	3.909327206507656E-03	4.645623146986024E-03	1.748004121006575E-03
2.469203694331862E-03	1.833276821033027E-03	2.120895513262196E-03	2.490098561063295E-03	9.134936739775175E-04
1.281866963495067E-03	1.836939968308109E-03	4.427319136728022E-04	4.949738182518414E-04	5.800913951796118E-04
2.099867389103362E-04	2.932384904461002E-04	4.097323592942723E-04	1.911369116931788E-04	1.997141016534249E-05
1.220047498181108E-03	3.457936034730549E-03	2.377640091189008E-03	1.570111112623541E-03	2.065553068158545E-03
4.332029836360598E-04	2.295567713567667E-03	3.012382211516784E-03	1.259252765701742E-03	9.148381667046676E-04
3.928189077432003E-03	5.039465502570407E-03	2.046611096309315E-03	2.963247882598756E-03	2.328127678000431E-03
2.190070440233655E-03	2.776244319538059E-03	1.104151814384261E-03	1.593659918560886E-03	2.437577691936250E-03
6.202418553580474E-04	5.221093088308858E-04	6.594344502649227E-04	2.590676561279608E-04	3.729963558228518E-04
5.591781367058243E-04	2.776572006752894E-04	3.032597978782719E-05	3.062079995434943E-04	9.681386746513642E-04
7.456082697848516E-04	4.578978092766076E-04	6.901769310797793E-04	1.560484108623459E-04	6.735614978577271E-04
1.014316290744267E-03	4.579536058914711E-04	3.359549643634826E-04	1.187579808050406E-03	1.761159923492610E-03
7.793418249234293E-04	1.141428370443757E-03	9.491034522826233E-04	6.772231074253788E-04	9.955197673597921E-04
4.338515735919416E-04	6.343522114284787E-04	1.034667244717720E-03	2.762978671430848E-04	1.643195323492281E-04
2.407727128301826E-04	1.039146620065646E-04	1.517219387255853E-04	2.437509554358946E-04	1.273375037266457E-04
1.452082383958461E-05	3.614955550166256E-05	1.240202113922246E-04	1.049062508323152E-04	6.142467995946791E-05
1.025402298509660E-04	2.468346115899797E-05	9.073469157764020E-05	1.514290151926310E-04	7.285023765032540E-05
5.375031013727738E-05	1.648178368583253E-04	2.723035003824597E-04	1.292030215927072E-04	1.904787051156355E-04
1.662402217627879E-04	9.610924729033017E-05	1.578119908510982E-04	7.405909460399417E-05	1.090781628126407E-04
1.877459398137415E-04	5.222674252969297E-05	2.372436092373203E-05	3.885897267916340E-05	1.810081914070725E-05
2.663695809866891E-05	4.532934510959655E-05	2.487908638262474E-05	2.922520813185860E-06	

CONVERGENCE VALUE 8.799901521990179E-16

3.487096805102190E-02	7.224666556883234E-02	3.773990185255553E-02	2.503097174684881E-02	2.638501132813850E-02
4.917254421652252E-03	2.785257547626866E-02	3.527686701798543E-02	1.384964818019092E-02	9.711758517706153E-03
3.688896745665824E-02	4.880236086683024E-02	2.019807098122966E-02	2.890784153320088E-02	2.259884610947497E-02
1.585764579390654E-02	2.230892397200652E-02	9.721679440172888E-03	1.401593424084338E-02	2.293797975710548E-02
6.057338008418827E-03	3.172152981014395E-03	4.564595521604898E-03	2.053909106178139E-03	2.967689332625135E-03
5.023853877036512E-03	2.737613690613740E-03	3.175592154526225E-04	9.286733066011881E-03	2.075928868234509E-02
1.076654088736206E-02	8.014407638563667E-03	7.900648593056009E-03	1.378072125781738E-03	1.069753573873299E-02
1.000699493314518E-02	3.301584542325330E-03	1.883607634474389E-03	1.659808506164934E-02	1.522523235608604E-02
4.832509669438196E-03	5.199712079550354E-03	3.354427086365835E-03	8.483531263869113E-03	7.743209732693310E-03
2.397917603322762E-03	2.453599366232399E-03	2.952627365713601E-03	5.920054868598706E-04	1.891168787154210E-03
1.742262669799395E-03	5.375794473723920E-04	5.348493002649915E-04	6.185377051255396E-04	2.311037002881240E-04
2.105292507800717E-05	6.555704198779611E-03	1.521520137693089E-02	7.876416671155358E-03	6.137482475131823E-03
5.886739970253005E-03	1.020340498408891E-03	8.737737710374217E-03	8.228423441498685E-03	2.793395431847514E-03
1.901371981460736E-03	1.408361415723609E-02	1.280890548136612E-02	4.121181262625957E-03	5.506350527419345E-03
3.756875229524915E-03	7.473660991298185E-03	6.704913393433103E-03	2.083995063229337E-03	2.742045153731168E-03
3.549962536381796E-03	7.907393498915918E-04	1.712220085619134E-03	1.537580969549627E-03	4.708506193007271E-04
6.133448330662254E-04	7.674186978564025E-04	3.259938588554421E-04	3.206929283054358E-05	1.624564834709919E-03
3.825047410510441E-03	1.948424971567217E-03	1.636818013478837E-03	1.541478062401855E-03	2.855277652916005E-04
2.375133217264640E-03	2.222273621127805E-03	8.167303445102537E-04	5.834989636485702E-04	3.973189078509463E-03
3.571750442114757E-03	1.251497207057781E-03	1.774933664912555E-03	1.286833869216635E-03	2.176323679976962E-03
1.921223172859526E-03	6.524523960104695E-04	9.196739090572007E-04	1.280396434079448E-03	3.053155643772704E-04
5.110178160229189E-04	4.492069949328316E-04	1.501032282505664E-04	2.106518862498917E-04	2.851419066650890E-04
1.312837212940293E-04	1.363642552692087E-05	1.411997827429457E-03	3.813052684167082E-03	2.439621066675103E-03
1.633791547841861E-03	2.004221514826539E-03	3.939639673307662E-04	2.362501721325424E-03	2.873962013636887E-03
1.122953398734767E-03	7.992725572918636E-04	3.909327206507311E-03	4.645623146985615E-03	1.748004121006421E-03
2.469203694331644E-03	1.833276821082866E-03	2.120895518262009E-03	2.490098561068076E-03	9.134936739774370E-04
1.281866963494954E-03	1.836939968307947E-03	4.427319136727632E-04	4.949738182617977E-04	5.800919951795606E-04
2.099367889103177E-04	2.932384904460743E-04	4.097323592942363E-04	1.911369116931620E-04	1.997141016534074E-05
1.220047498181001E-03	3.437936034730241E-03	2.377640091188798E-03	1.570111112623403E-03	2.065553068158363E-03
4.332029836360215E-04	2.295567713567464E-03	3.012332211516518E-03	1.259252765701631E-03	9.143381667045869E-04
3.928189077431661E-03	5.039465502569963E-03	2.046611096309134E-03	2.963247882598495E-03	2.328127678000226E-03
2.190070440233472E-03	2.776244319537815E-03	1.104151814384164E-03	1.593659918660745E-03	2.437577691936036E-03
6.202418553579928E-04	5.221098088308398E-04	6.594344502648645E-04	2.590676561279380E-04	3.729963558228189E-04
5.591731367057750E-04	2.776572006752650E-04	3.032697978782453E-05	3.062079995434673E-04	9.681336746512789E-04
7.458082697847858E-04	4.578978092765672E-04	6.901769310797184E-04	1.560484108623321E-04	6.735614978576679E-04
1.014316290744178E-03	4.579536058914308E-04	3.359549643634530E-04	1.187579808050304E-03	1.761159923492455E-03
7.793418249233608E-04	1.141423870438657E-03	9.491034522825399E-04	6.772231074253192E-04	9.955197678597044E-04
4.238515735919034E-04	6.343522114284229E-04	1.034557244717629E-03	2.762978671430605E-04	1.643195328492136E-04
2.407727128361615E-04	1.039146620065555E-04	1.517219387255719E-04	2.437509554358732E-04	1.278375037266345E-04
1.452062383958334E-05	3.614955550165938E-05	1.240202113922137E-04	1.049062508823059E-04	6.142467995948249E-05
1.025402298509570E-04	2.463346115899580E-05	9.073469157763221E-05	1.514290151926177E-04	7.285023765031900E-05
5.375031013727265E-05	1.643178368583107E-04	2.723035003824357E-04	1.292030215926958E-04	1.904787051156187E-04
1.662402217627733E-04	9.610924729082170E-05	1.578119908510843E-04	7.405909460398766E-05	1.090781628126311E-04
1.877459398137250E-04	5.222674252968838E-05	2.372436092372995E-05	3.865897267915999E-05	1.810081914070567E-05
2.663695809866657E-05	4.532934510959257E-05	2.487908638262255E-05	2.922520813185604E-06	

CONVERGENCE VALUE 8.854177275667866E-16

3.487096805101902E-02	7.224665556882639E-02	3.773990185255243E-02	2.503097174684675E-02	2.638501132813633E-02
4.917254421651848E-03	2.785257547626637E-02	3.527636701798253E-02	1.384864818018978E-02	9.711758517705356E-03
3.688896745665521E-02	4.880236086682623E-02	2.019807098122800E-02	2.890784153319850E-02	2.259884610947311E-02
1.585764579390523E-02	2.230692397200468E-02	9.721679440172092E-03	1.401593424084222E-02	2.293797975710359E-02
6.057338003418328E-03	3.172152981014134E-03	4.564595521304518E-03	2.053909106177970E-03	2.967689332624889E-03
5.023853377036099E-03	2.737613690613515E-03	3.175592164525963E-04	9.286733066011117E-03	2.075923868234338E-02
1.076664088736117E-02	3.014407638563006E-03	7.900648593055358E-03	1.378072125781625E-03	1.069753573873211E-02
1.000699493314435E-02	3.301584542326558E-03	1.883669634474233E-03	1.659608506164797E-02	1.522523235606478E-02
4.832509669487797E-03	5.199712079549925E-03	3.354427086365558E-03	8.483531263868413E-03	7.743209732692669E-03
2.397917603322564E-03	2.453599366232197E-03	2.952627365713357E-03	5.920054863598216E-04	1.891168787154054E-03
1.742262669799252E-03	5.375794473728476E-04	5.348493002649474E-04	6.185377051254836E-04	2.311037002881049E-04
2.105292507800543E-05	6.555704198779071E-03	1.521520137692964E-02	7.876416671154707E-03	6.137482475131318E-03
5.886739970252520E-03	1.020340498408307E-03	8.737737710373495E-03	8.228423441498005E-03	2.793395431647283E-03
1.901871981460579E-03	1.408361415723493E-02	1.280890548136506E-02	4.121181262625616E-03	5.506350527418839E-03
3.756875229524604E-03	7.473660991297567E-03	6.704913393432551E-03	2.063995063229165E-03	2.742045153730941E-03
3.549962536381503E-03	7.907393498915263E-04	1.712220085618993E-03	1.537580969549500E-03	4.708506193006882E-04
6.133448330661746E-04	7.674186978563389E-04	3.259938588554151E-04	3.206929283054092E-05	1.624564834709785E-03
3.825047410510124E-03	1.948424971567055E-03	1.636818013478702E-03	1.541478062401728E-03	2.855277652915770E-04
2.375133217264444E-03	2.222273621127621E-03	8.167303445101861E-04	5.834989686485219E-04	3.973189078509136E-03
3.571750442114461E-03	1.251497207057678E-03	1.774933664912408E-03	1.286833869216529E-03	2.176323679976782E-03
1.921223172859367E-03	6.524523960104154E-04	9.196739090571246E-04	1.280396434079342E-03	3.053155643772451E-04
5.110178160223767E-04	4.492069949327944E-04	1.501032282505540E-04	2.106518862498743E-04	2.851419066650654E-04
1.312837212940184E-04	1.353642552691974E-05	1.411997827429340E-03	3.813052684166767E-03	2.439621066674901E-03
1.633791547841726E-03	2.004221514826423E-03	3.939639673307336E-04	2.362501721325229E-03	2.878962018636648E-03
1.122953393734674E-03	7.992725572917974E-04	3.909327206506988E-03	4.645623146985231E-03	1.748004121006276E-03
2.469203694331440E-03	1.833276821082714E-03	2.120895518261834E-03	2.490098561067870E-03	9.134936739773614E-04
1.291866963494848E-03	1.836939968307795E-03	4.427319136727265E-04	4.949738182617569E-04	5.800919951795127E-04
2.099867889103003E-04	2.932384904460501E-04	4.097323592942024E-04	1.911369116931462E-04	1.997141016533906E-05
1.220047493130900E-03	3.467936034729953E-03	2.377640091188601E-03	1.570111112623273E-03	2.065553068158192E-03
4.332029836359857E-04	2.295567713567274E-03	3.012382211516269E-03	1.259252765701527E-03	9.148381667045112E-04
3.928139077431336E-03	5.039465502569546E-03	2.046611096308965E-03	2.963247832598250E-03	2.328127678000033E-03
2.190070440233290E-03	2.776244319537585E-03	1.104151814384072E-03	1.593559913560613E-03	2.437577691935834E-03
6.202418553579415E-04	5.221098088307966E-04	6.594344502648101E-04	2.590676561279156E-04	3.729963558227880E-04
5.591781367057287E-04	2.776572006752420E-04	3.032697978782202E-05	3.062079995434420E-04	9.681386746511987E-04
7.458032697847240E-04	4.578973092763293E-04	6.901769310796613E-04	1.560484103623192E-04	6.735614978576121E-04
1.014316290744094E-03	4.579536058913929E-04	3.359549643634252E-04	1.187579808050205E-03	1.761159923492309E-03
7.793418249232961E-04	1.141428870443562E-03	9.491034522824612E-04	6.772231074252632E-04	9.955197678596220E-04
4.338515735918674E-04	6.343522114283703E-04	1.034667244717543E-03	2.762978671430377E-04	1.643195328492000E-04
2.407727123361415E-04	1.039146620065409E-04	1.517219387255594E-04	2.437509554358530E-04	1.278375037266239E-04
1.452082383958213E-05	3.614955550165639E-05	1.240202113922034E-04	1.049062508322972E-04	6.142467995947741E-05
1.025402298509485E-04	2.468346115399376E-05	9.073469157762469E-05	1.514290151926052E-04	7.285023765031296E-05
5.375031013726819E-05	1.648178368532971E-04	2.723035003824132E-04	1.292030215926851E-04	1.904787051156029E-04
1.662402217627595E-04	9.610924729081374E-05	1.578119908510712E-04	7.405909460398152E-05	1.090781628126221E-04
1.877459398137094E-04	5.222674252968406E-05	2.372436092372798E-05	3.885897267915677E-05	1.810081914070417E-05
2.663695809866436E-05	4.532934510958882E-05	2.487908638262049E-05	2.922520813185362E-06	

IN STMT 22 PROGRAM RETURNS FROM MAIN PROCEDURE.

APPENDIX B

ANALYSIS OF THE PROBABILITIES

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$PLC TIME=05,ATR,XREF

*OPTIONS IN EFFECT*    TIME=(5.0),PAGES=30,LINES=1500,ATR,XREF,FLAGW,NOBNDRY,NOCMNTS,SORMGIN=(2,72,1),ERRORS=(3,50),
*OPTIONS IN EFFECT*    TABSIZE=21732,UDEF,SOURCE,OPLIST,NOCMPRS,HDRPG,AUXIO=10000,LINECT=60,NOALIST,MCALL,MTEXT,DUMP=(
*OPTIONS IN EFFECT*    S,F,L,E,U,R),DUMPE=(S,F,L,E,U,R),DUMPT=(S,F,L,E,U,R)

 ANALYS: PROCEDURE OPTIONS(MAIN);                                    PL/C-R7,1--68 05/16/77 10:12 PAGE   1

 STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

    1                        ANALYS: PROCEDURE OPTIONS(MAIN);
                             /*
                             LIST OF VARIABLES:
                                  PROCEDURES:
                                     ANALYS     THE DRIVER PROGRAM.
                                     COMPUTE2   THE FUNCTION WHICH RETURNS THE QUOTIENT OF A GIVEN
                                                PROBABILITY AND THE EXTERNAL NODE COUNT.
                                     PRINT      THE ROUTINE TO OUTPUT THE RESULTS IN THE FREQUENCY
                                                ANALYSIS.
                                     PROBAL     THE ROUTINE TO COMPUTE THE PROBABILITY OF A SPLIT.
                                     REMAP1     THE ROUTINE TO DETERMINE THE LEFT AND RIGHT BRANCHES
                                                AND THE EXTERNAL NODE COUNT OF A GIVEN STATE WHICH
                                                HAS A ONE KEY THIRD LEVEL GRANDPARENT.
                                     REMAP2     THE ROUTINE TO COMPUTE THE LEFT, RIGHT, AND MIDDLE
                                                BRANCHES IN A GIVEN STATE WHICH HAS A TWO KEY THIRD
                                                LEVEL NODE.
                                     UTILIZZ    THE ROUTINE TO COMPUTE THE UTILIZATION AND THE PERFORM
                                                THE FREQUENCY ANALYSIS.
                                  ARRAYS:
                                     ETAB       THE EXTERNAL NODE COUNT FROM A GIVEN TWO LEVEL, STATE
                                                NODE.
                                     EXNCS      THE EXTERNAL NODES DESCENDING FROM TWO KEY FIRST LEVEL
                                                NODES FOR A GIVEN TWO LEVEL STATE.
                                     FSTLVL     THE ARRAY TO CONTAIN THE FREQUENCY OF NODES AT THE
                                                FIRST LEVEL WITH ONE AND TWO KEYS DESCENDING FROM
                                                INDICATED SECOND AND THIRD LEVEL NODES.
                                     LRCMB      THE LEFT AND RIGHT COMBINATIONS.
                                     PRCB       THE ASYMPTOTIC PROBABILITIES FROM THE PREVIOUS PROGRAM.
                                     SECLVL     THE ARRAY TO CONTAIN THE FREQUENCY OF SECOND LEVEL
                                                NODES DESCENDING FROM INDICATED THIRD LEVEL NODES.
                                     STATE1     THE NUMBER OF FIRST LEVEL NODES WITH ONE KEY DESCENDING
                                                FROM A GIVEN SECOND LEVEL STATE.
                                     STATE2     THE NUMBER OF SECOND LEVEL NODES WITH TWO KEYS
                                                DESCENDING FROM A GIVEN SECOND LEVEL STATE.
                                  SCALARS:
                                     CRTNCS     THE NUMBER OF CRITICAL(TWO KEY) NODES.
                                     ECCLNT     THE EXTERNAL NODE COUNT.
                                     F1         THE FREQUENCY OF STATE ONE EXTERNAL NODES.
                                     F1FRME     THE NORMALIZED FREQUENCY OF STATE ONE EXTERNAL NODES.
                                     F2         THE FREQUENCY OF STATE TWO EXTERNAL NODES.
                                     F2FRME     THE NORMALIZED FREQUENCY OF STATE TWO EXTERNAL NODES.
                                     I,J,K      INDICES.
                                     L          THE LEFT BRANCH.
                                     LR         THE LEFT-RIGHT BRANCH PAIR.
                                     M          THE MIDDLE BRANCH.
                                     PROBSPT    THE PROBABILITY OF A SPLIT.
                                     R          THE RIGHT BRANCH.
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STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

                                     RL       THE LEFT-RIGHT BRANCH PAIR.
                                     SUM      A VARIABLE USED IN SUMATIONS.
                                     UTAL     THE UTILIZATION.
                                  */
  2    1          1               DCL PROB(224) FLOAT(16) STATIC;
  3    1          1               DCL(SUM,ECOUNT,PROBSPT,CRTNDS) FLOAT(16);
  4    1          1               DCL ETAB(0:7) FLOAT(16) STATIC INITIAL(0.0E0,4.0E0,5.0E0,6.0E0,6.0E0,
                                    7.0E0,8.0E0,9.0E0);
  5    1          1               DCL(I,M,L,R,LR,RL) FIXED BINARY;
  6    1          1               DCL EXNDS (7) FLOAT(16) STATIC INITIAL(0.0E0,3.0E0,6.0E0,0.0E0,3.0E0,
                                  6.0E0,9.0E0);
  7    1          1               DCL LRCMB(28,2) FIXED BINARY STATIC INITIAL (
                                    1,1, 2,1, 2,2, 3,1, 3,2, 3,3, 4,1, 4,2, 4,3, 4,4,
                                    5,1, 5,2, 5,3, 5,4, 5,5, 6,1, 6,2, 6,3, 6,4, 6,5, 6,6,
                                    7,1, 7,2, 7,3, 7,4, 7,5, 7,6, 7,7);
                                  /*
                                   NORMALIZE THE INITIAL PROBABILITIES
                                  */
  8    1          1               GET LIST(PROB);
  9    1          1                SUM = 0.0E0;
 10    1          1                DO I=1 TO 224;
 11    1     1    1                  SUM = PROB(I)+SUM;
 12    1     1    1                END;
 13    1          1                DO I=1 TO 224;
 14    1     1    1                  PROB(I) = PROB(I)/SUM;
 15    1     1    1                END;
                                  /*
                                  THE DRIVER PROGRAM CALLS THE ROUTINE TO COMPUTE THE PROBABILITY OF A
                                  SPLIT.
                                  */
 16    1          1               CALL PROBAL;
                                  /*
                                  IT ALSO CALLS THE ROUTINE TO COMPUTE THE UTILIZATION.
                                  */
 17    1          1               CALL UTILIZZ;
 18    1          1               RETURN;
                                  /*
                                  THIS IS THE ROUTINE TO COMPUTE THE PROBABILITY OF A SPLIT.
                                  */
 19    1          1               PROBAL:PROCEDURE;
 20    2          2                PROBSPT = 0.0E0;
                                  /*
                                   THE PROBABILITY OF A SPLIT AT THE THIRD LEVEL IS COMPUTED FROM THE TWO
                                   KEY THIRD LEVEL NODES.
                                  */
 21    2          2                DO I=29 TO 224;
 22    2     1    2                  CALL REMAP2(I,M,L,R,ECOUNT);
 23    2     1    2                  CRTNDS = 0.0E0;
                                  /*
                                     SUM THE CRITICAL NODES.
                                  */
 24    2     1    2                  IF M>4 THEN CRTNDS = CRTNDS + EXNDS(M);
 26    2     1    2                  IF L>4 THEN CRTNDS = CRTNDS + EXNDS(L);
 28    2     1    2                  IF R>4 THEN CRTNDS = CRTNDS + EXNDS(R);
 30    2     1    2                  PROBSPT = PROB(I)*CRTNDS/ECOUNT + PROBSPT;
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 STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

   31    2    1     2          END;
   32    2          2         PUT SKIP EDIT('THE RESULTS OF THE PROBRABILITY ANALYSIS ')(A);
   33    2          2         PUT SKIP ;
   34    2          2         PUT SKIP EDIT('THE PROBABLITY OF A LEVEL THREE SPLIT   ',PROBSPT)(X(5),
                              A,E(25,15));
                              /*
                                 DETERMINE THE STATE TWO PROBABILITY OF A SPLIT
                              */
   35    2          2         PROBSPT = 0.0E0;
   36    2          2         DO I=1 TO 224;
   37    2    1     2           CRTNDS = 0.0E0;
   38    2    1     2           IF I<29 THEN
   39    2    1     2             DO;
   40    2    2     2               CALL REMAP1(I,L,R,ECOUNT);
                              /*
                                    SUM THE CRITICAL NODES.
                              */
   41    2    2     2               IF L>4 THEN CRTNDS = EXNDS(L);
   43    2    2     2               IF R>4 THEN CRTNDS = EXNDS(R) + CRTNDS;
   45    2    2     2             END;
   46    2    1     2           ELSE DO;
   47    2    2     2             CALL REMAP2(I,M,L,R,ECOUNT);
   48    2    2     2             IF M>4 THEN CRTNDS = EXNDS(M);
                              /*
                                 SUM THE CRITICAL NODES.
                              */
   50    2    2     2             IF L>4 THEN CRTNDS = EXNDS(L) + CRTNDS;
   52    2    2     2             IF R>4 THEN CRTNDS = EXNDS(R) + CRTNDS;
   54    2    2     2           END;
   55    2    1     2           PROBSPT = PROB(I)*CRTNDS/ECOUNT + PROBSPT;
   56    2    1     2         END;
   57    2          2         PUT SKIP EDIT('THE PROBABLITY OF A LEVEL TWO SPLIT   ',PROBSPT)(X(5),A,
                              E(25,15));
                              /*
                              DETERMINE THE STATE ONE PROBABILITY OF A SPLIT
                              */
   58    2          2         PROBSPT = 0.0E0;
   59    2          2         DO I=1 TO 224 ;
   60    2    1     2           CRTNDS = 0.0E0;
   61    2    1     2           IF I<29 THEN
   62    2    1     2             DO;
   63    2    2     2               CALL REMAP1(I,L,R,ECOUNT);
   64    2    2     2               CRTNDS = EXNDS(L) + EXNDS(R);
   65    2    2     2             END;
   66    2    1     2           ELSE DO;
   67    2    2     2             CALL REMAP2(I,M,L,R,ECOUNT);
   68    2    2     2             CRTNDS = EXNDS(M) + EXNDS(L) + EXNDS(R);
   69    2    2     2           END;
   70    2    1     2           PROBSPT = PROB(I)*CRTNDS/ECOUNT + PROBSPT;
   71    2    1     2         END;
   72    2          2         PUT SKIP EDIT('THE PROBABLITY OF A LEVEL ONE SPLIT   ',PROBSPT)(X(5),A,
                              E(25,15));
   73    2          2         RETURN;
   74    2          2         END PROBAL;
                              /*
```

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 STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

                                 THIS ROUTINE COMPUTES THE UTILIZATION AND PERFORMS THE FREQUENCY ANALYSIS.
                                 */
   75    1             1         UTILIZZ:PROCEDURE;
   76    2             3         DCL SECLVL(2,2) FLOAT(16) INITIAL((4)0.0E0);
   77    2             3         DCL FSTLVL(2,2,2) FLOAT(16) INITIAL((8)0.0E0);
   78    2             3         DCL STATE2(7)  FLOAT(16) STATIC INITIAL(0.0E0,1.0E0,2.0E0,0.0E0,1.0E0,
                                  2.0E0,3.0E0);
   79    2             3         DCL STATE1(7) FLOAT(16) STATIC INITIAL(2.0E0,1.0E0,0.0E0,3.0E0,2.0E0,
                                  1.0E0,0.0E0);
   80    2             3         DCL COMPUTE2 ENTRY(FIXED BINARY, FIXED BINARY,FLOAT(16))
                                 RETURNS(FLOAT(16));
   81    2             3         DCL(F1,F2,F1PRME,F2PRME,UTAL,DIFF) FLOAT(16);
   82    2             3         PUT SKIP(2);
   83    2             3         PUT SKIP EDIT ('THE RESULTS OF THE ANALYSIS OF THE UTILIZATION')(A);
   84    2             3         PUT SKIP;
                                 /*
                                 COMPUTE THE UTILIZATION AT LEVEL THREE.
                                 */
   85    2             3         F2 = 0.0E0;
   86    2             3         F1 = 0.0E0;
   87    2             3         DO I=1 TO 224;
   88    2     1       3           IF I>28 THEN DO;
   90    2     2       3             CALL REMAP2(I,M,L,R,ECOUNT);
   91    2     2       3             F2 = PROB(I)/ECOUNT + F2;
   92    2     2       3           END;
   93    2     1       3           ELSE DO;
   94    2     2       3             CALL REMAP1(I,L,R,ECOUNT);
   95    2     2       3             F1 = PROB(I)/ECOUNT + F1;
   96    2     2       3            END;
   97    2     1       3         END;
                                 /*
                                 DETERMINE THE NORMALIZED FREQUENCIES.
                                 */
   98    2             3         F1PRME = F1/(F1 + F2);
   99    2             3         F2PRME = F2/(F1 + F2);
  100    2             3         UTAL = (F1PRME + 2*F2PRME)/2;
  101    2             3         PUT SKIP EDIT('THE UTILIZATION AT LEVEL THREE    ',UTAL)(X(5),A,
                                 E(25,15));
                                 /*
                                 COMPUTE THE UTILIZATION AT LEVEL TWO.
                                 */
  102    2             3         F1 = 0.0E0;
  103    2             3         F2 = 0.0E0;
  104    2             3         M = 0 ;
  105    2             3         DO I=1 TO 224;
  106    2     1       3           IF I<29 THEN CALL REMAP1(I,L,R,ECOUNT);
  108    2     1       3           ELSE DO;
  109    2     2       3             CALL REMAP2(I,M,L,R,ECOUNT);
  110    2     2       3             IF M>3 THEN F2 = PROB(I)/ECOUNT + F2;
  112    2     2       3             ELSE F1 = PROB(I)/ECOUNT + F1;
  113    2     2       3           END;
  114    2     1       3           IF L>3 THEN F2 = PROB(I)/ECOUNT + F2;
  115    2     1       3           ELSE F1 = PROB(I)/ECOUNT + F1;
  117    2     1       3           IF R>3 THEN F2 = PROB(I)/ECOUNT + F2;
  119    2     1       3           ELSE F1 = PROB(I)/ECOUNT + F1;
```

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 STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

  120    2     1      3          END;
                                 /*
                                 NORMALIZE THE FREQUENCIES.
                                 */
  121    2            3          F1PRME = F1/(F1 + F2);
  122    2            3          F2PRME = F2/(F1 + F2);
  123    2            3          UTAL = (F1PRME + 2*F2PRME)/2;
  124    2            3          PUT SKIP EDIT('THE UTILIZATION AT LEVEL TWO   ',UTAL)(X(5),A,E(25,15));
                                 /*
                                 DETERMINE THE UTILIZATION AT LEVEL ONE.
                                 */
  125    2            3          F1 = 0.0E0;
  126    2            3          F2 = 0.0E0;
  127    2            3          M = 0 ;
  128    2            3          DO I=1 TO 224;
  129    2     1      3            IF I<29 THEN CALL REMAP1(I,L,R,ECOUNT);
  131    2     1      3            ELSE DO;
  132    2     2      3              CALL REMAP2(I,M,L,R,ECOUNT);
  133    2     2      3              F1 = STATE1(M)*PROB(I)/ECOUNT + F1;
  134    2     2      3              F2 = STATE2(M)*PROB(I)/ECOUNT + F2;
  135    2     2      3            END;
                                 /*
                                 DETERMINE THE RELATIVE FREQUENCIES FOR STATES ONE AND TWO.
                                 */
  136    2     1      3          F1 = STATE1(L)*PROB(I)/ECOUNT + F1;
  137    2     1      3          F2 = STATE2(L)*PROB(I)/ECOUNT + F2;
  138    2     1      3          F1 = STATE1(R)*PROB(I)/ECOUNT + F1;
  139    2     1      3          F2 = STATE2(R)*PROB(I)/ECOUNT + F2;
  140    2     1      3          END;
  141    2            3          F1PRME = F1/(F1 + F2);
  142    2            3          F2PRME = F2/(F1 + F2);
  143    2            3          UTAL = (F1PRME + 2*F2PRME)/2;
  144    2            3          PUT SKIP EDIT('THE UTILIZATION AT LEVEL ONE   ',UTAL)(X(5),A,E(25,15));
  145    2            3          PUT PAGE;
                                 /*
                                 THIS IS THE ROUTINE TO ASCERTAIN THE FREQUENCIES OF NODES DESCENDING
                                 FROM ONE AND TWO KEY PARENTS AND GRANDPARENTS.  IT DECOMPOSES EACH STATE
                                 AND PROPORTIONS THE FREQUENCIES OF THIS STATE OCCURRING FROM THE
                                 ASYMPTOTIC STATE PROBABILITIES.
                                 */
  146    2            3          SECLVL = 0.0E0;
  147    2            3          DO I=1 TO 224;
  148    2     1      3            IF I<29 THEN DO;
  150    2     2      3              J=1;
  151    2     2      3              CALL REMAP1(I,L,R,ECOUNT);
  152    2     2      3              M=0;
  153    2     2      3            END;
  154    2     1      3            ELSE DO;
  155    2     2      3            J=2;
  156    2     2      3              CALL REMAP2(I,M,L,R,ECOUNT);
  157    2     2      3            END;
  158    2     1      3            IF L>3 THEN K=2;
  160    2     1      3            ELSE K=1;
  161    2     1      3            SECLVL(J,K) = SECLVL(J,K) + PROB(I) /ECOUNT;
  162    2     1      3            FSTLVL(J,K,1) = FSTLVL(J,K,1) + PROB(I)*STATE1(L)/ECOUNT;
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STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

 163    2    1     3              FSTLVL(J,K,2) = FSTLVL(J,K,2) + PROB(I)*STATE2(L)/ECOUNT;
 164    2    1     3              IF R>3 THEN K=2;
 166    2    1     3              ELSE K=1;
 167    2    1     3              SECLVL(J,K) = SECLVL(J,K) + PROB(I) /ECOUNT;
 168    2    1     3              FSTLVL(J,K,1) = FSTLVL(J,K,1) + PROB(I)*STATE1(R)/ECOUNT;
 169    2    1     3              FSTLVL(J,K,2) = FSTLVL(J,K,2) + PROB(I)*STATE2(R)/ECOUNT;
 170    2    1     3              IF M¬=0 THEN DO;
 172    2    2     3              IF M>3 THEN K=2;
 174    2    2     3              ELSE K=1;
 175    2    2     3              SECLVL(J,K) = SECLVL(J,K) + PROB(I) /ECOUNT;
 176    2    2     3              FSTLVL(J,K,1) = FSTLVL(J,K,1) + PROB(I)*STATE1(M)/ECOUNT;
 177    2    2     3              FSTLVL(J,K,2) = FSTLVL(J,K,2) + PROB(I)*STATE2(M)/ECOUNT;
 178    2    2     3              END;
 179    2    1     3            END;
 180    2          3            PUT SKIP LIST('INTERNAL FREQUENCY ANALYSIS');
 181    2          3            CALL PRINT;
 182    2          3            SUM = 0.0E0;  ECOUNT = 0.0E0;
                                /*
                                THE FREQUENCIES ARE NORMALIZED.
                                */
 184    2          3            DO J=1 TO 2;
 185    2    1     3              DO K=1 TO 2;
 186    2    2     3                SUM = SUM + SECLVL(J,K);
 187    2    2     3                DO L=1 TO 2;
 188    2    3     3                  ECOUNT = ECOUNT + FSTLVL(J,K,L);
 189    2    3     3                END;
 190    2    2     3              END;
 191    2    1     3            END;
 192    2          3            DO J=1 TO 2;
 193    2    1     3              DO K=1 TO 2;
 194    2    2     3                DO L=1 TO 2;
 195    2    3     3                  FSTLVL(J,K,L) = FSTLVL(J,K,L)/ECOUNT;
 196    2    3     3                END;
 197    2    2     3                SECLVL(J,K) = SECLVL(J,K)/SUM;
 198    2    2     3              END;
 199    2    1     3            END;
 200    2          3            PUT SKIP;
 201    2          3            PUT SKIP LIST('AFTER NORMALIZING ');
 202    2          3            CALL PRINT;
                                /*
                                THE FREQUENCIES ARE NORMALIZED BY PAIRS DESCENDING FROM A GIVEN NODE.
                                */
 203    2          3            DO J=1 TO 2;
 204    2    1     3              SUM = SECLVL(J,1) + SECLVL(J,2);
 205    2    1     3              SECLVL(J,1) = SECLVL(J,1)/SUM;
 206    2    1     3              SECLVL(J,2) = SECLVL(J,2)/SUM;
 207    2    1     3              DO K=1 TO 2;
 208    2    2     3                ECOUNT = FSTLVL(J,K,1) + FSTLVL(J,K,2);
 209    2    2     3                FSTLVL(J,K,1) = FSTLVL(J,K,1)/ECOUNT;
 210    2    2     3                FSTLVL(J,K,2) = FSTLVL(J,K,2)/ECOUNT;
 211    2    2     3              END;
 212    2    1     3            END;
 213    2          3            PUT SKIP(2);
 214    2          3            PUT SKIP LIST('AFTER NORMALIZING PAIRS');
 215    2          3            CALL PRINT;
```

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STMT LEVEL NEST BLOCK MLVL  SOURCE TEXT

 216    2          3          RETURN;
                              /*
                              THIS ROUTINE PRINTS THE RESULTS OF THE FREQUENCY ANALYSIS.
                              */
 217    2          3          PRINT:PROCEDURE;
 218    3          4          PUT SKIP;
 219    3          4          PUT SKIP EDIT('DESCENDING FROM A ONE KEY THIRD LEVEL NODE')(X(5),A);
 220    3          4          PUT SKIP EDIT('HAVING ONE KEY ',SECLVL(1,1),' HAVING TWO KEYS ',
                              SECLVL(1,2))(X(10),A(15),E(25,15),A(17),E(25,15));
 221    3          4          PUT SKIP EDIT('DESCENDING FROM A TWO KEY THIRD LEVEL NODE')(X(5),A);
 222    3          4          PUT SKIP EDIT('HAVING ONE KEY ',SECLVL(2,1),' HAVING TWO KEYS ',
                              SECLVL(2,2))(X(10),A(15),E(25,15),A(17),E(25,15));
 223    3          4          PUT SKIP EDIT('DESCENDING FROM A 3 LVL 1 KEY, AND 2 LEVEL 1 KEY')(X(5),
                              A);
 224    3          4          PUT SKIP EDIT('HAVING ONE KEY ',FSTLVL(1,1,1),' HAVING TWO KEYS ',
                              FSTLVL(1,1,2))(X(10),A(15),E(25,15),A(17),E(25,15));
 225    3          4          PUT SKIP EDIT('DESCENDING FROM A 3 LVL 1 KEY, AND 2 LEVEL 2 KEY')
                              (X(5),A);
 226    3          4          PUT SKIP EDIT('HAVING ONE KEY ',FSTLVL(1,2,1),' HAVING TWO KEYS ',
                              FSTLVL(1,2,2))(X(10),A(15),E(25,15),A(17),E(25,15));
 227    3          4          PUT SKIP EDIT('DESCENDING FROM A 3 LVL 2 KEY, AND 2 LEVEL 1 KEY')
                              (X(5),A);
 228    3          4          PUT SKIP EDIT('HAVING ONE KEY ',FSTLVL(2,1,1),' HAVING TWO KEYS ',
                              FSTLVL(2,1,2))(X(10),A(15),E(25,15),A(17),E(25,15));
 229    3          4          PUT SKIP EDIT('DESCENDING FROM A 3 LVL 2 KEY, AND 2 LEVEL 2 KEY')
                              (X(5),A);
 230    3          4          PUT SKIP EDIT('HAVING ONE KEY ',FSTLVL(2,2,1),' HAVING TWO KEYS ',
                              FSTLVL(2,2,2))(X(10),A(15),E(25,15),A(17),E(25,15));
 231    3          4          RETURN;
 232    3          4          END;
 233    2          3          COMPUTE2:PROCEDURE(I,K,ECOUNT) RETURNS (FLOAT);
 234    3          5          DCL(I,K) FIXED BINARY;
 235    3          5          DCL ECOUNT FLOAT(16);
 236    3          5            RETURN(PROB(I)/ECOUNT);
 237    3          5          END COMPUTE2;
 238    2          3          END UTILIZZ;
 239    1          1          REMAP1: PROCEDURE(I,L,R,ECOUNT);
 240    2          6          DCL(I,L,R) FIXED BINARY;
 241    2          6          DCL ECOUNT FLOAT(16);
 242    2          6            L = LRCNE(I,1);
 243    2          6            R = LRCNE(I,2);
 244    2          6            ECOUNT = ETAB(L) + ETAB(R);
 245    2          6          RETURN;
 246    2          6          END;
 247    1          1          REMAP2: PROCEDURE(I,M,L,R,ECOUNT);
 248    2          7          DCL (I,M,L,R,LR ) FIXED BINARY;
 249    2          7          DCL ECOUNT FLOAT(16);
 250    2          7            LR = MOD(I-1,28) + 1;
 251    2          7            L = LRCNE(LR,1);
 252    2          7            R = LRCNE(LR,2);
 253    2          7            M = (I-1)/28;
 254    2          7            ECOUNT = ETAB(M) + ETAB(L) + ETAB(R);
 255    2          7          RETURN;
 256    2          7          END;
 257    1          1          END ANALYS;
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THE RESULTS OF THE PROBABILITY ANALYSIS

     THE PROBABLITY OF A LEVEL THREE SPLIT      7.745252506188077E-02
     THE PROBABLITY OF A LEVEL TWO SPLIT      1.820798398216139E-01
     THE PROBABLITY OF A LEVEL ONE SPLIT      4.285714285733760E-01

THE RESULTS OF THE ANALYSIS OF THE UTILIZATION

     THE UTILIZATION AT LEVEL THREE      6.754285727844964E-01
     THE UTILIZATION AT LEVEL TWO      5.758777614144300E-01
     THE UTILIZATION AT LEVEL ONE      6.666666666075493E-01
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INTERNAL FREQUENCY ANALYSIS

     DESCENDING FROM A ONE KEY THIRD LEVEL NODE
          HAVING ONE KEY      6.350298044722853E-02 HAVING TWO KEYS      3.705259626991754E-02
     DESCENDING FROM A TWO KEY THIRD LEVEL NODE
          HAVING ONE KEY      5.416521044037350E-02 HAVING TWO KEYS      2.735915266204903E-02
     DESCENDING FROM A 3 LVL 1 KEY, AND 2 LEVEL 1 KEY
          HAVING ONE KEY      8.119504751169567E-02 HAVING TWO KEYS      4.581071334276118E-02
     DESCENDING FROM A 3 LVL 1 KEY, AND 2 LEVEL 2 KEY
          HAVING ONE KEY      7.028201755650799E-02 HAVING TWO KEYS      3.487577125324458E-02
     DESCENDING FROM A 3 LVL 2 KEY, AND 2 LEVEL 1 KEY
          HAVING ONE KEY      7.197727134625374E-02 HAVING TWO KEYS      3.635314953449288E-02
     DESCENDING FROM A 3 LVL 2 KEY, AND 2 LEVEL 2 KEY
          HAVING ONE KEY      5.025994939865367E-02 HAVING TWO KEYS      2.581750868729364E-02

AFTER NORMALIZING

     DESCENDING FROM A ONE KEY THIRD LEVEL NODE
          HAVING ONE KEY      3.487539296554093E-01 HAVING TWO KEYS      2.034964239129092E-01
     DESCENDING FROM A TWO KEY THIRD LEVEL NODE
          HAVING ONE KEY      2.974805475117307E-01 HAVING TWO KEYS      1.502590989155508E-01
     DESCENDING FROM A 3 LVL 1 KEY, AND 2 LEVEL 1 KEY
          HAVING ONE KEY      1.894551108607673E-01 HAVING TWO KEYS      1.068916645598571E-01
     DESCENDING FROM A 3 LVL 1 KEY, AND 2 LEVEL 2 KEY
          HAVING ONE KEY      1.779913742586536E-01 HAVING TWO KEYS      8.137679959096573E-02
     DESCENDING FROM A 3 LVL 2 KEY, AND 2 LEVEL 1 KEY
          HAVING ONE KEY      1.579459684747194E-01 HAVING TWO KEYS      8.482401558054770E-02
     DESCENDING FROM A 3 LVL 2 KEY, AND 2 LEVEL 2 KEY
          HAVING ONE KEY      1.312732150307581E-01 HAVING TWO KEYS      6.024085360373085E-02

AFTER NORMALIZING PAIRS

     DESCENDING FROM A ONE KEY THIRD LEVEL NODE
          HAVING ONE KEY      6.315203531690091E-01 HAVING TWO KEYS      3.684791468309908E-01
     DESCENDING FROM A TWO KEY THIRD LEVEL NODE
          HAVING ONE KEY      6.644051959329438E-01 HAVING TWO KEYS      3.355948040670561E-01
     DESCENDING FROM A 3 LVL 1 KEY, AND 2 LEVEL 1 KEY
          HAVING ONE KEY      6.393020831485722E-01 HAVING TWO KEYS      3.606979168514277E-01
     DESCENDING FROM A 3 LVL 1 KEY, AND 2 LEVEL 2 KEY
          HAVING ONE KEY      6.862498649290808E-01 HAVING TWO KEYS      3.137501350709191E-01
     DESCENDING FROM A 3 LVL 2 KEY, AND 2 LEVEL 1 KEY
          HAVING ONE KEY      6.644234441356826E-01 HAVING TWO KEYS      3.355765558643173E-01
     DESCENDING FROM A 3 LVL 2 KEY, AND 2 LEVEL 2 KEY
          HAVING ONE KEY      6.854494605370087E-01 HAVING TWO KEYS      3.145505394629913E-01

IN STMT   18  PROGRAM RETURNS FROM MAIN PROCEDURE.
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VITA[2]

John Henry Ray

Candidate for the Degree of

Master of Science

Thesis: A THREE LEVEL ASYMPTOTIC ANALYSIS OF MINIMUM ORDER B-TREES

Major Field: Computing and Information Science

Biographical:

Personal Data: Born in Duncan, Oklahoma, September 24, 1950, the son of Mr. and Mrs. D. L. Ray.

Education: Graduated from Duncan High School, Duncan, Oklahoma, in May, 1968; received the Bachelor of Arts degree from Cameron University, Lawton, Oklahoma, in May, 1975, with a major in Mathematics; completed requirements for the Master of Science degree at Oklahoma State University, Stillwater, Oklahoma, in July, 1977.

Professional Experience: Maintenance Supervisor for the City of Duncan, Oklahoma, September, 1971 to January, 1973.